A First Course in Digital Communications

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October 4, 2006

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Sampling and Quantization

The previous two chapters have reviewed and discussed the various characteristics of signals along with methods of describing and representing them. This chapter begins the discussion of transmitting the signals or messages by a digital communication system. Though one can easily visualize messages produced by sources that are inherently digital in nature, witness text messaging via the keyboard or keypad, two of the most common message sources, audio and video, are analog, i.e., they produce continuous time signals. To make them amenable for digital transmission it is first required to transform the analog information source into digital symbols which are compatible with digital processing and transmission.

The first step in this transformation process is to discretize the time axis which involves sampling the continuous time signal at discrete values of time. The sampling process, primarily how many samples per second are needed to exactly represent the signal, practical sampling schemes, and how to reconstruct, at the receiver, the analog message from the samples is discussed first. This is followed by a brief discussion of three pulse modulation techniques, sort of a halfway house between the analog modulation methods of AM and FM and the various modulation/demodulation methods which are the focus of the rest of the text.

Though time has been discretized by the sampling process the sample values are still analog, i.e., they are continuous variables. To represent the sample value by a digital symbol chosen from a finite set necessitates the choice of a discrete set of amplitudes to represent the continuous range of possible amplitudes. This process is known as quantization and unlike discretization of the time axis, it results in a distortion of the original signal since it is a many-to-one mapping. The measure of this distortion is commonly expressed by the signal power to quantization noise power ratio, SNR_q. Various approaches to quantization and the resultant SNR_q are the major focus of this chapter.

The final step is to map (or encode) the quantized signal sample into a string of digital, typically binary, symbols, commonly known as pulse code modulation (PCM). The complete process of analog-to-digital (A/D) conversion is a special, but important, case of source coding.†

4.1 Sampling of Continuous-Time Signals

The first operation in A/D conversion is the sampling process. Both the theoretical and practical implementations of this process are studied in this section.

4.1.1 Ideal (or Impulse) Sampling

The sampling process converts an analog waveform into a sequence of discrete samples that are usually spaced *uniformly* in time. This process can be mathematically described as in Figure 4.1-(a). Here the analog waveform $m(t)$ is multiplied by a periodic train of unit *impulse* functions $s(t)$ to produced the sampled waveform $m_s(t)$. The expression for $s(t)$ is as follows:

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \tag{4.1}$$

Thus the sampled waveform $m_s(t)$ can be expressed as,

$$\begin{aligned} m_s(t) = m(t)s(t) &= \sum_{n=-\infty}^{\infty} m(t)\delta(t - nT_s) \\ &= \sum_{n=-\infty}^{\infty} m(nT_s)\delta(t - nT_s) \end{aligned} \tag{4.2}$$

The parameter T_s in (4.1) and (4.2) is the period of the impulse train, also referred to as the *sampling period*. The inverse of the sampling period, $f_s = 1/T_s$, is called the sampling frequency or *sampling rate*. Figures 4.1-(b) to 4.1-(d) graphically illustrate the ideal sampling process. It is intuitive that the higher the sampling rate is, the more accurate

† A more general source coding process not only involves A/D conversion but also some form of data compression to remove the redundancy of the information source.

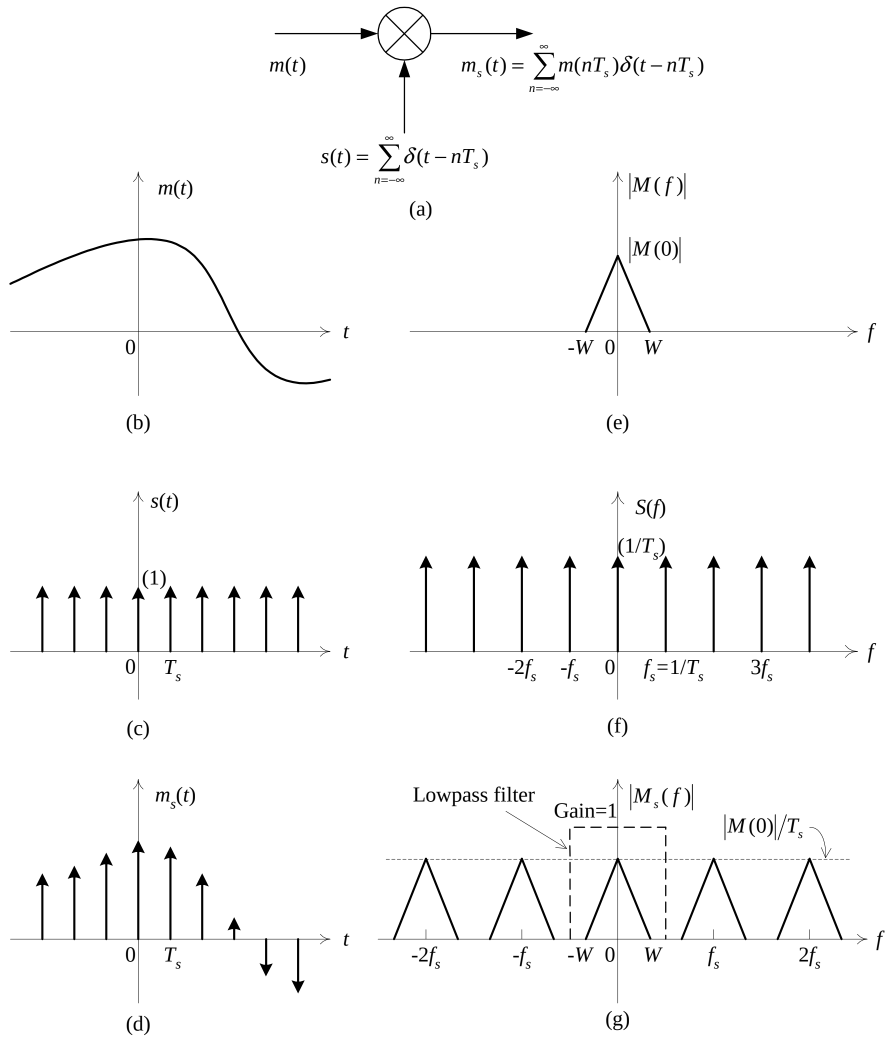


Fig. 4.1. Ideal sampling process: (a) Mathematical model, (b) Analog signal, (c) Impulse train, (d) Sampled version of the analog signal, (e) Spectrum of bandlimited signal, (f) Spectrum of the impulse train, (g) Spectrum of sampled waveform.

the representation of $m(t)$ by $m_s(t)$ is. However, to achieve a high efficiency, it is desired to use as a low sampling rate as possible. Thus an important question is: what is the minimum sampling rate for the sampled version $m_s(t)$ to exactly represent the original analog signal $m(t)$?

For the family of band-limited signals, this question is answered by the *sampling theorem*, which is derived next.

Consider the Fourier transform of the sampled waveform $m_s(t)$. Since $m_s(t)$ is the product of $m(t)$ and $s(t)$, the Fourier transform of $m_s(t)$ is the *convolution* of the Fourier transforms of $m(t)$ and $s(t)$. Recall that the Fourier transform of an impulse train is another impulse train, where the values of the periods of the two trains are reciprocally related to one another. The Fourier transform of $s(t)$ is given by

$$S(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \tag{4.3}$$

Also note that convolution with an impulse function simply shifts the original function as follows:

$$X(f) \circledast \delta(f - f_0) = X(f - f_0) \tag{4.4}$$

From the above equations, the transform of the sampled waveform can be written as,

$$\begin{aligned} M_s(f) = M(f) \circledast S(f) &= M(f) \circledast \left[\frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \right] \\ &= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} M(f - nf_s) \end{aligned} \tag{4.5}$$

Equation (4.5) shows that the spectrum of the sampled waveform consists of an infinite number of scaled and shifted copies of the spectrum of the original signal $m(t)$. More precisely, the spectrum $M(f)$ is scaled by $1/T_s$ and periodically repeated every f_s. It should be noted that the relation in (4.5) holds for any continuous-time signal $m(t)$, even if it is not band-limited, or of finite energy.

However, for the band-limited waveform $m(t)$ with bandwidth limited to W Hertz, generic Fourier transform of the sampled signal $m_s(t)$ is illustrated in Figure 4.1. The triangular shape chosen for the magnitude spectrum of $m(t)$ is only for ease of illustration. In general, $M(f)$ can be of arbitrary shape, as long as it is confined to $[0, W]$. Since within the original bandwidth (around zero frequency) the spectrum of the sampled waveform is the same at that of the original signal (except for a scaling factor $1/T_s$), it suggests that the original waveform $m(t)$ can be completely recovered from $m_s(t)$ by an ideal low-pass filter of bandwidth W as shown in Figure 4.1-(g). However, a closer investigation of Figure 4.1-(g) reveals that this is only possible if the sampling rate f_s is high enough so that there is no overlap among the copies of $M(f)$ in the spectrum of $m_s(t)$. It is easy to see that the condition for no overlapping of the copies of $M(f)$ is $f_s \geq 2W$, therefore the minimum sampling rate is $f_s = 2W$. When the sampling rate $f_s < 2W$ (under-sampling), then the copies of $M(f)$ will overlap in the frequency domain and it is not possible to recover the original signal $m(t)$ by filtering. The distortion of the recovered signal due to under-sampling is referred to as *aliasing*.

It has been shown in the frequency domain that the original continuous signal $m(t)$ can be completely recovered from the sampled signal $m_s(t)$. Next we wish to show how to reconstruct the continuous signal $m(t)$ from it sampled values $m(nT_s)$, $n = 0, \pm 1, \pm 2, \ldots$. To this end, write the Fourier transform of $m_s(t)$ as follows.

$$\begin{aligned} M_s(f) = \mathcal{F}\{m_s(t)\} &= \sum_{n=-\infty}^{\infty} m(nT_s)\mathcal{F}\{\delta(t - nT_s)\} \\ &= \sum_{n=-\infty}^{\infty} m(nT_s)\exp(-j2\pi nfT_s) \qquad (4.6) \end{aligned}$$

Since $M(f) = M_s(f)/f_s$, for $-W \leq f \leq W$, one can write

$$M(f) = \frac{1}{f_s}\sum_{n=-\infty}^{\infty} m(nT_s)\exp(-j2\pi nfT_s), \quad -W \leq f \leq W \qquad (4.7)$$

The signal $m(t)$ is the inverse Fourier transform of $M(f)$ and it can be found as follows:

$$\begin{aligned} m(t) &= \mathcal{F}^{-1}\{M(f)\} = \int_{-\infty}^{\infty} M(f)\exp(j2\pi ft)\mathrm{d}f \\ &= \int_{-W}^{W} \frac{1}{f_s}\sum_{n=-\infty}^{\infty} m(nT_s)\exp(-j2\pi nfT_s)\exp(j2\pi ft)\mathrm{d}f \\ &= \frac{1}{f_s}\sum_{n=-\infty}^{\infty} m(nT_s)\int_{-W}^{W}\exp[j2\pi f(t - nT_s)]\mathrm{d}f \\ &= \sum_{n=-\infty}^{\infty} m(nT_s)\frac{\sin[2\pi W(t - nT_s)]}{\pi f_s(t - nT_s)} \\ &= \sum_{n=-\infty}^{\infty} m\left(\frac{n}{2W}\right)\frac{\sin(2\pi Wt - n\pi)}{(2\pi Wt - n\pi)} \qquad (4.8) \\ &= \sum_{n=-\infty}^{\infty} m\left(\frac{n}{2W}\right)\mathrm{sinc}(2Wt - n) \qquad (4.9) \end{aligned}$$

where, to arrive at the last two equations, the minimum sampling rate $f_s = 2W$ has been used and $\text{sinc}(x) \equiv \sin(\pi x)/(\pi x)$.

Equation (4.9) provides an *interpolation formula* for the construction of the original signal $m(t)$ from its sampled values $m(n/2W)$. The sinc function $\text{sinc}(2Wt)$ plays the role of an *interpolating function* (also known as the sampling function). In essence, each sample is multiplied by a delayed version of the interpolating function and all the resulting waveforms are added up to obtain the original signal.

Now the sampling theorem can be stated as follows.

Theorem 4.1 (Sampling Theorem). *A signal having no frequency components above W Hertz is completely described by specifying the values of the signal at periodic time instants that are separated by at most $1/2W$ seconds.*

The theorem stated in terms of the sampling rate, $f_s \geq 2W$, is known as the *Nyquist criterion*. The sampling rate $f_s = 2W$ is called the *Nyquist rate* with the reciprocal called the *Nyquist interval*.

The sampling process considered so far is known as *ideal sampling* because it involves ideal impulse functions. Obviously, ideal sampling is not practical. In the next two sections two practical methods to implement sampling of continuous-time signals are introduced.

4.1.2 Natural Sampling

Figure 4.2-(a) shows the mathematical model of natural sampling. Here, the analog signal $m(t)$ is multiplied by the pulse train, or switching waveform $p(t)$ shown in Figure 4.2-(c). Let $h(t) = 1$ for $0 \leq t \leq \tau$ and $h(t) = 0$ otherwise. Then the pulse train $p(t)$ can be written as,

$$p(t) = \sum_{n=-\infty}^{\infty} h(t - nT_s) \tag{4.10}$$

Natural sampling is therefore very simple to implement since it requires only an on/off gate. Figure 4.2-(d) shows the resulting sampled waveform, which is simply $m_s(t) = m(t)p(t)$. As in ideal sampling, here the sampling rate f_s also equals to the inverse of the period T_s of the pulse train, i.e., $f_s = 1/T_s$. Next it is shown that with natural sampling, a bandlimited waveform can also be reconstructed from its sampled version as long as the sampling rate satisfies the Nyquist criterion. As before, the analysis is carried out in frequency domain.

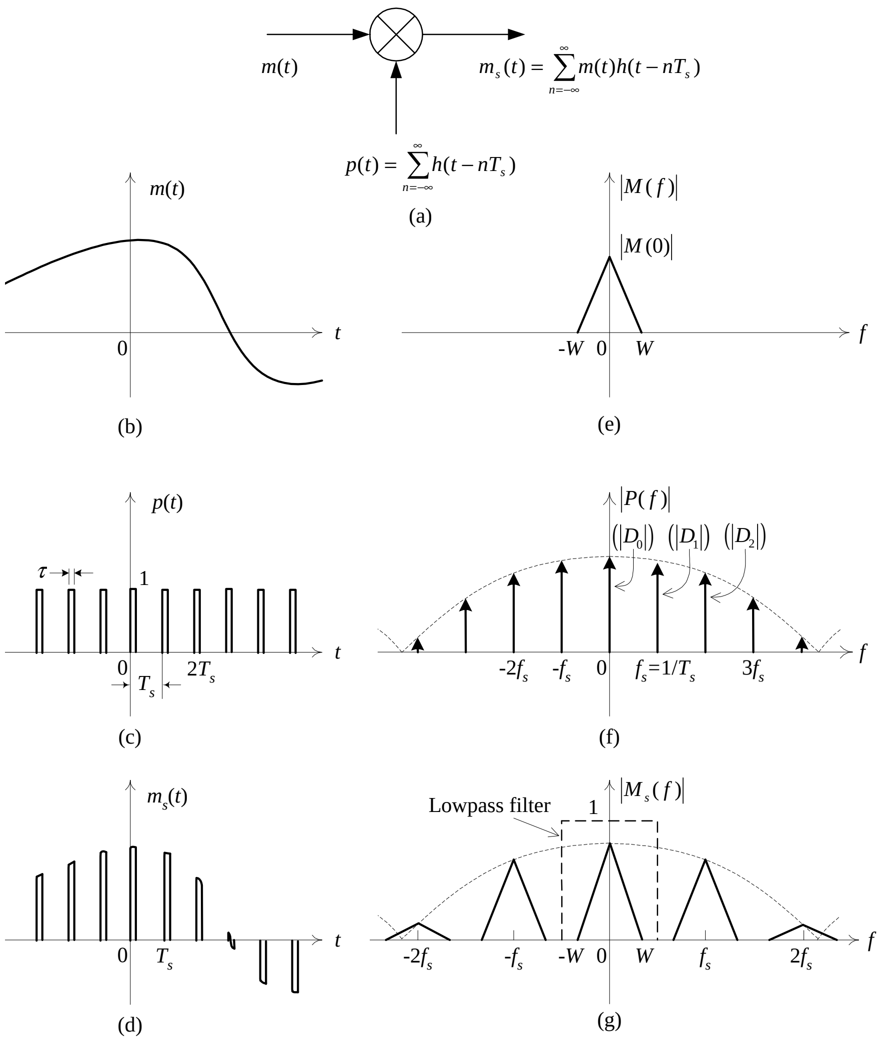


Fig. 4.2. The natural sampling process: (a) Mathematical model, (b) Analog signal, (c) Rectangular pulse train, (d) Sampled version of the analog signal, (e) Spectrum of bandlimited signal, (f) Spectrum of the pulse train, (g) Spectrum of sampled waveform.

Recall that the periodic pulse train $p(t)$ can be expressed in a Fourier series as follows:

$$p(t) = \sum_{n=-\infty}^{\infty} D_n \exp(j2\pi n f_s t) \tag{4.11}$$

where D_n is the Fourier coefficient, given by

$$D_n = \frac{\tau}{T_s} \operatorname{sinc}\left(\frac{n\tau}{T_s}\right) \mathrm{e}^{-j\pi n\tau/T_s} \tag{4.12}$$

Thus the sampled waveform is

$$m_s(t) = m(t) \sum_{n=-\infty}^{\infty} D_n \exp(j2\pi n f_s t) \tag{4.13}$$

The Fourier transform of $m_s(t)$ can be found as follows:

$$\begin{aligned} M_s(f) = \mathcal{F}\{m_s(t)\} &= \sum_{n=-\infty}^{\infty} D_n \mathcal{F}\{m(t)\exp(j2\pi n f_s t)\} \\ &= \sum_{n=-\infty}^{\infty} D_n M(f - n f_s) \end{aligned} \tag{4.14}$$

Similar to the ideal sampling case, Equation (4.14) shows that $M_s(f)$ consists of an infinite number of copies of $M(f)$, which are periodically shifted in frequency every f_s Hertz. However, here the copies of $M(f)$ are not uniformly weighted (scaled) as in the ideal sampling case, but rather they are weighted by the Fourier series coefficients of the pulse train. The spectrum of the sampled waveform $m_s(t)$ is shown in Figure 4.2-(g) where $m(t)$ is again a bandlimited waveform. It can be seen from 4.2-(g) that, despite the above difference, the original signal $m(t)$ can be equally well reconstructed using a lowpass filter as long as the Nyquist criterion is satisfied . Finally, it should be noted that natural sampling can be considered to be a practical approximation of ideal sampling, where an ideal impulse is approximated by a narrow rectangular pulse. With this perspective, it is not surprising that when the width τ of the pulse train approaches zero, the spectrum in (4.14) converges to Equation (4.5).

4.1.3 Flat-Top Sampling

Flat-top sampling is the most popular sampling method. This sampling process involves two simple operations:

- Instantaneous sampling of the analog signal $m(t)$ every T_s seconds. As in the ideal and natural sampling cases, it will be shown that to reconstruct the original signal $m(t)$ from its sampled version, the sampling rate $f_s = 1/T_s$ must satisfy the Nyquist criterion.
- Maintain the value of each sample for a duration of τ seconds.

In circuit technology, these two operations are referred to as *sample and hold*. The flat-top sampled waveform is illustrated in Figure 4.3

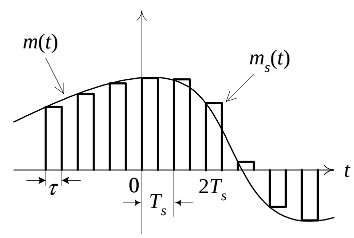


Fig. 4.3. Flat-top sampling: $m(t)$ is the original analog signal and $m_s(t)$ is the sampled signal.

It is straightforward to verify that the flat-top sampling described above can be mathematically modeled as shown in Figure 4.4-(a). Note that Figure 4.4-(a) is an extension of Figure 4.1-(a), where a filter with impulse response $h(t)$ is added at the end.

Again, we are interested in the spectrum of the sampled signal $m_s(t)$, which is related to the original signal $m(t)$ through the following expression:

$$m_s(t) = \left[m(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right] \circledast h(t) \tag{4.15}$$

The Fourier transform of $m_s(t)$ is determined as follows:

$$\begin{aligned} M_s(f) &= \mathcal{F}\left\{ m(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right\} \mathcal{F}\{h(t)\} \\ &= \frac{1}{T_s} H(f) \sum_{n=-\infty}^{\infty} M(f - nf_s) \end{aligned} \tag{4.16}$$

where $H(f)$ is the Fourier transform of the rectangular pulse $h(t)$, given by $H(f) = \tau \text{sinc}(f\tau)\exp(-j\pi f\tau)$.

Equations (4.16) and (4.5) imply that the spectrum of the signal produced by flat-top sampling is essentially the spectral of the signal pro-

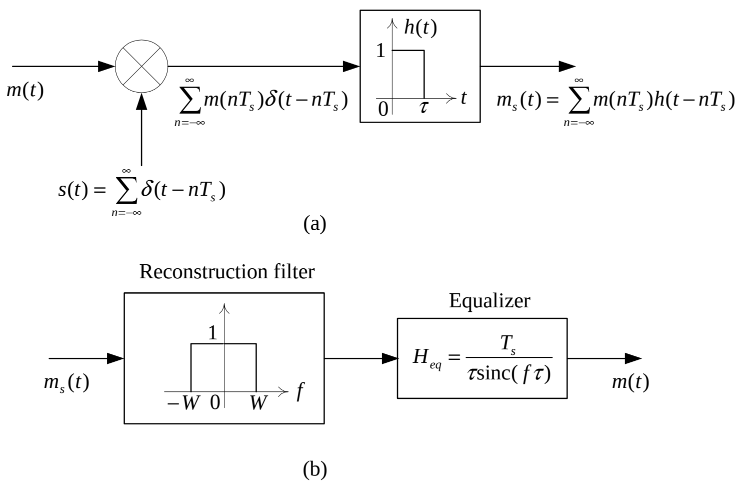


Fig. 4.4. The flat-top sampling process: (a) Sampling, (b) Reconstruction.

duced by ideal sampling shaped by $H(f)$. Since $H(f)$ has the form of a sinc function, each spectral component of the ideal-sampled signal is weighted differently, hence causing amplitude distortion. As a consequence of this distortion, it is not possible to reconstruct the original signal using an lowpass filter, even when the Nyquist criterion is satisfied. This is illustrated in Figure 4.5.

In fact, if the Nyquist criterion is satisfied, then passing the flat-top sampled signal through a lowpass filter (with a bandwidth of W) produces the signal whose Fourier transform is $(1/T_s)M(f)H(f)$. Thus the distortion due to $H(f)$ can be corrected by connecting an *equalizer* in cascade with the lowpass reconstruction filter, as shown in Figure 4.4-(b). Ideally, the amplitude response of the equalizer is given by

$$|H_{\text{eq}}| = \frac{T_s}{|H(f)|} = \frac{T_s}{\tau \text{sinc}(f\tau)} \tag{4.17}$$

Finally, it should be noted that for a duty cycle $\tau/T_s \leq 0.1$, the amplitude distortion is less than 0.5%. In this case, the need for equalization may be omitted in practical applications.

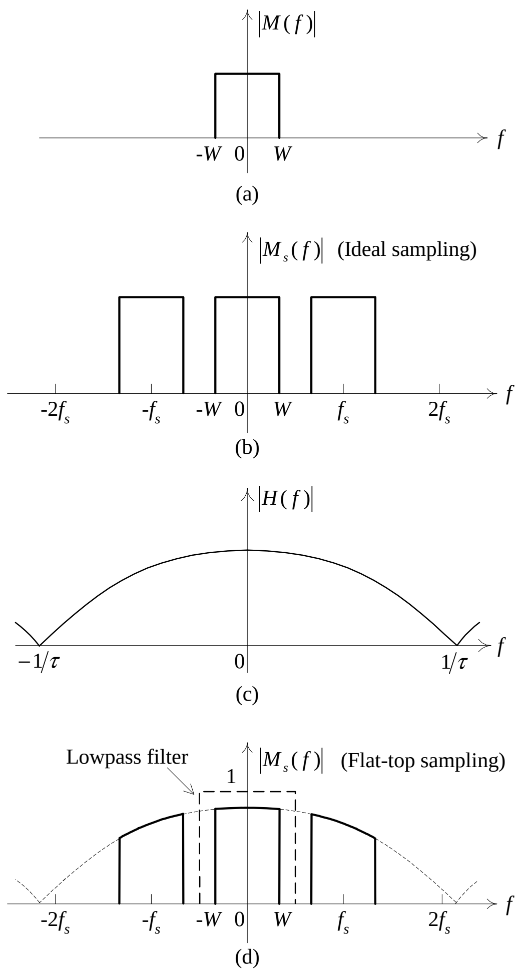


Fig. 4.5. Various spectrums in flat-top sampling process: (a) Magnitude spectrum of the original message, (b) Spectrum of the ideal sampled waveform, (c) $|H(f)| = |\tau \text{sinc}(f\tau)|$, (d) Spectrum of the flat-top sampled waveform.

4.2 Pulse Modulation

Recall that in analog (continuous wave) modulation, some parameter of a sinusoidal carrier $A_c \cos(2\pi f_c t + \theta)$ such as the amplitude A_c, the frequency f_c, or the phase θ is varied *continuously* in accordance with the message signal $m(t)$. Similarly, in pulse modulation, some parameter of a *pulse train* is varied in accordance with the sample values of a message signal.

Pulse amplitude modulation (PAM) is the simplest and most basis form of analog pulse modulation. In PAM, the *amplitudes* of regularly spaced pulses are varied in proportion to the corresponding sample values of a continuous message signal. In general, the pulses can be of some appropriate shape. In the simplest case, when the pulse is rectangular, then the PAM signal is identical to the signal produced by flat-top sampling described in Section 4.1.3.

It should be noted that PAM transmission does not improve the noise performance over base-band modulation, (which is the transmission of the original continuous signal). The main (perhaps the only) advantage of PAM is that it allows multiplexing, i.e., sharing the same transmission media by different sources (or users). This is because a PAM signal only occurs in slots of time, leaving the idle time for the transmission of other PAM signals. However, this advantage comes from the expense of larger transmission bandwidth, as can be seen from Figures 4.5-(a) and 4.5-(d).

It is well known that in analog frequency modulation (FM), bandwidth can be traded for noise performance. As mentioned before, PAM signals require larger transmission bandwidth without improving noise performance. This suggests that there should be better pulse modulations than PAM in terms of noise performance. Two such forms of pulse modulation are:

- Pulse width modulation (PWM): In PWM, the samples of the message signal are used to vary the *width* of the individual pulses in the pulse train.
- Pulse position modulation (PPM): In PPM, the position of a pulse relative to its original time of occurrence is varied in accordance with the sample values of the message.

Examples of PWM and PPM waveforms are shown in Figure 4.6 for a sinusoidal message.

Note that in PWM, long pulses (corresponding to large sample values) expend considerable power, while bearing no additional information. In fact, if only time transitions are preserved, then PWM becomes PPM. Accordingly, PPM is a more power-efficient form of pulse modulation than PWM.

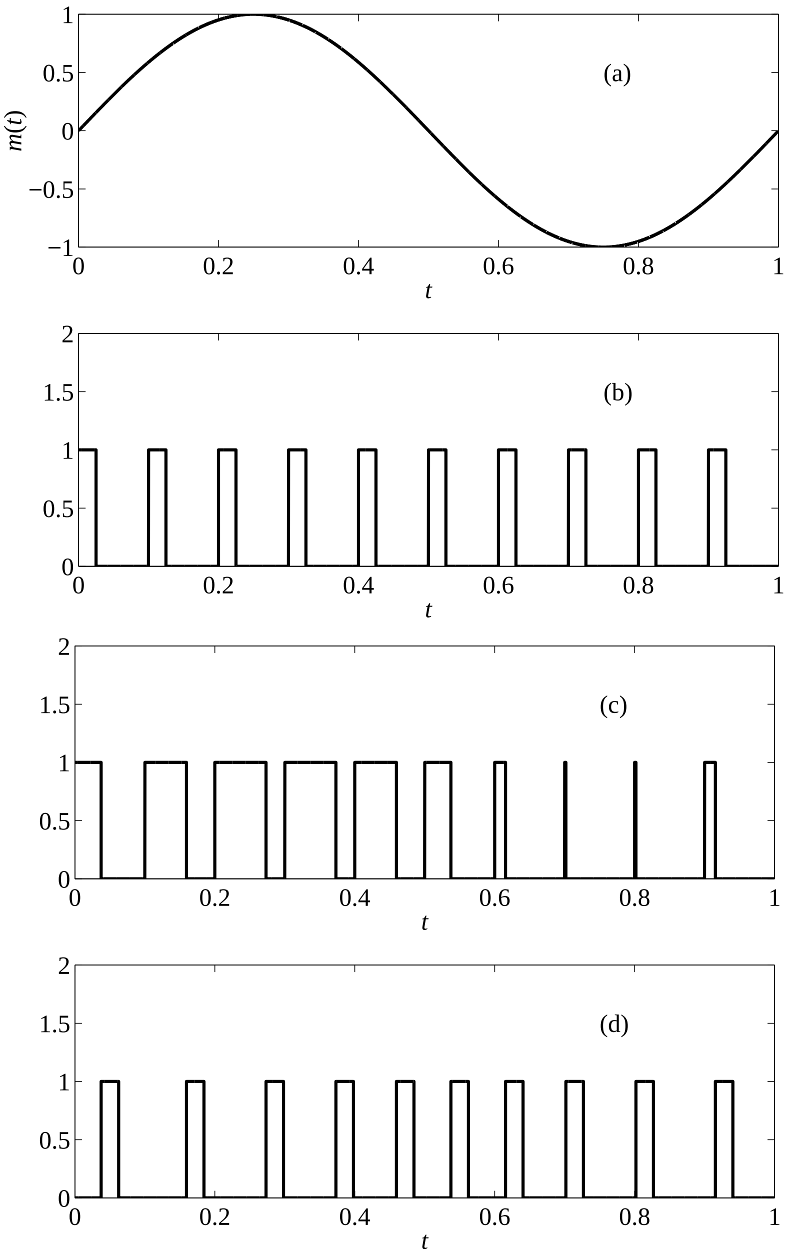


Fig. 4.6. Examples of PWM and PPM: (a) A Sinusoidal message, (b) Pulse carrier, (c) PWM waveform, (d) PPM waveform.

Regarding the noise performance of PWM and PPM systems, since the transmitted information (the sample values) is contained in the relative positions of the modulated pulses, the additive noise, which mainly introduces amplitude distortion, has much lesser effect. As a consequence, both PWM and PPM systems have better noise performance than PAM.

Pulse modulation techniques, however, are still an analog modulation. For digital communications of an analog source, one needs to proceed to the next step, i.e., quantization.

4.3 Quantization

In all the sampling processes described in the previous section, the sampled signals are discrete in time but still continuous in amplitude. To fully obtain a digital representation of a continuous signal, two further operations are needed: *quantization* of the amplitude of the sampled signal and *encoding* of the quantized values, as illustrated in Figure 4.7. This section discusses quantization.

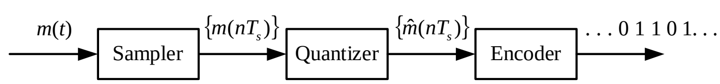


Fig. 4.7. Quantization and encoding oprations.

By definition, amplitude quantization is the process of transforming the sample amplitude $m(nT_s)$ of a message signal $m(t)$ at time $t = nT_s$ into a discrete amplitude $\hat{m}(nT_s)$ taken from a *finite* set of possible amplitudes. Clearly, if the finite set of amplitudes is chosen such that the spacing between two adjacent amplitude levels is sufficiently small, then the approximated (or quantized) signal, $\hat{m}(nT_s)$, can be made practically indistinguishable from the continuous sampled signal, $m(nT_s)$. Nevertheless, unlike the sampling process, there is always a loss of information associated with the quantization process, no matter how fine one may choose the finite set of the amplitudes for quantization. This implies that it is not possible to *completely* recover the sampled signal from the quantized signal.

In this section, we shall assume that the quantization process is *memoryless* and *instantaneous*, meaning that the quantization of sample value at time $t = nT_s$ is independent of earlier or later samples. With this assumption, the quantization process can be described as in Figure 4.8. Let the amplitude range of the continuous signal be partitioned into L intervals, where the lth interval, denoted by $\mathcal{I}_l$, is determined by the *decision levels* (also called the *threshold levels*) D_l and D_{l+1}:

$$\mathcal{I}_l : \{D_l < m \leq D_{l+1}\}, \quad l = 1, \ldots, L \tag{4.18}$$

Then the quantizer represents all the signal amplitudes in the interval $\mathcal{I}_l$ by some amplitude $T_l \in \mathcal{I}_l$ referred to as the *target level* (also known as the *representation level* or *reconstruction level*). The spacing between two adjacent decision levels is called a *step-size*. If the step-size is the same for each interval then the quantizer is called a *uniform quantizer*, otherwise the quantizer is non-uniform. The uniform quantizer is the simplest and most practical one. Besides having equal decision intervals the target level is chosen to lie in the middle of the interval.

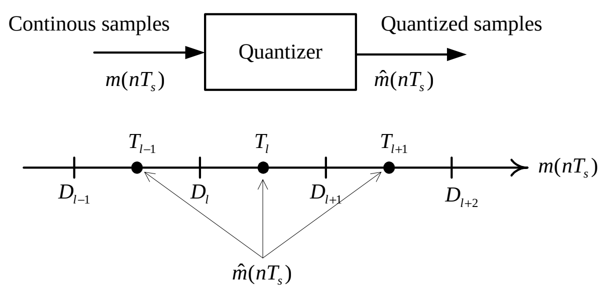


Fig. 4.8. Description of a memoryless quantizer.

4.3.1 Uniform Quantizer

From the description of the quantizer, it follows that the input/output characteristic of the quantizer (or *quantizer characteristic*) is a staircase function. Figures 4.9-(a) and 4.9-(b) display two uniform quantizer characteristics, called *midtread* and *midrise*. As can be seen from these figures, the classification whether a characteristic is midtread or midrise depends on whether the origin lies in the middle of a tread, or a rise of the staircase characteristic. For both characteristics, the decision levels are equally spaced and the lth target level is the midpoint of the lth interval, i.e.,

$$T_l = \frac{D_l + D_{l+1}}{2} \tag{4.19}$$

As an example, Figure 4.10 plots the input and output waveforms of a midrise uniform quantizer.

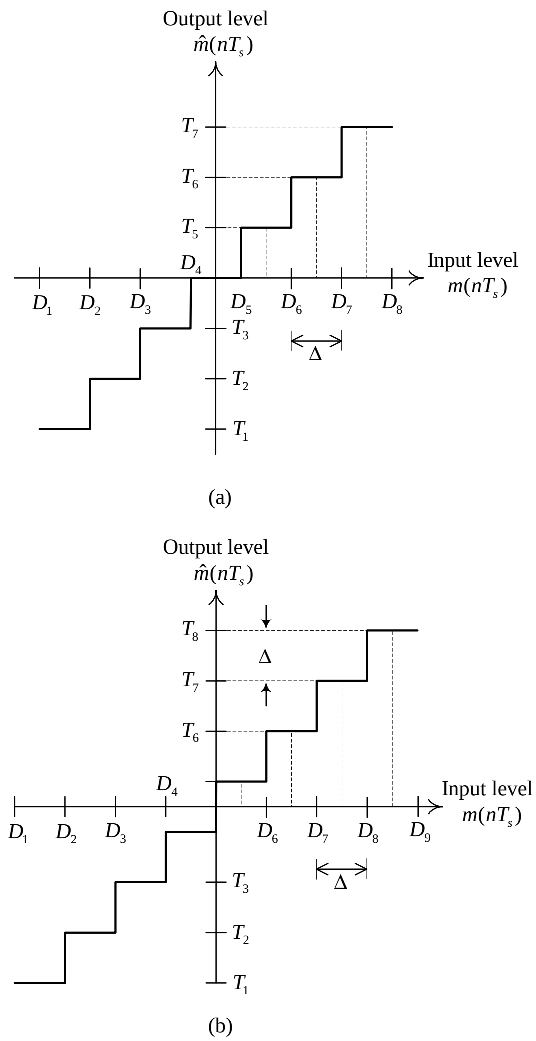


Fig. 4.9. Two types of uniform quantization: (a) midtread and (b) midrise.

As mentioned before, the quantization process always introduces an error. The performance of an quantizer is usually evaluated in terms of

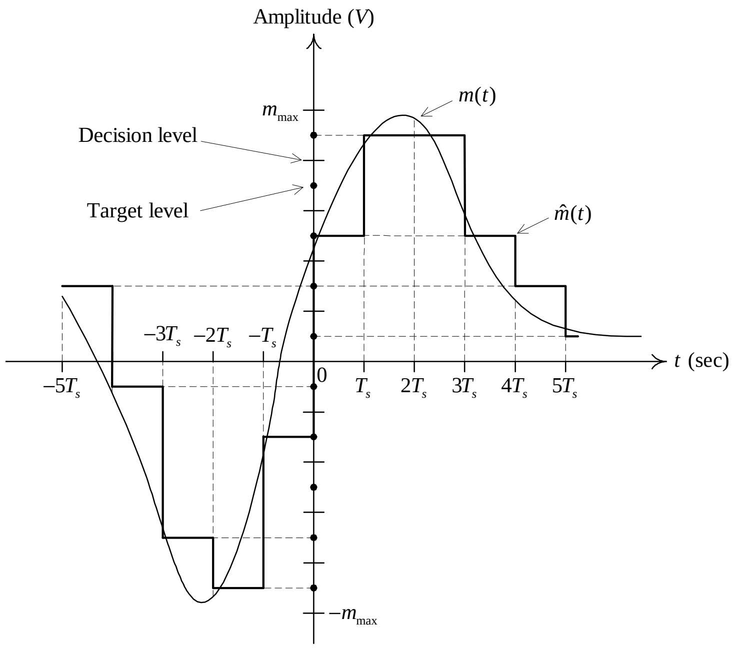


Fig. 4.10. An example of input and output of a midrise uniform quantizer.

its signal-to-quantization noise ratio. In what follows, this parameter is derived for the uniform quantizer.

Since we concentrate only on memoryless quantization, we can ignore the time index and simply write m and $\hat{m}$ instead of $m(nT_s)$ and $\hat{m}(nT_s)$ for the input and output of the quantizer respectively. Typically, the input m of the quantizer can be modeled as a zero-mean random variable with some probability density function (pdf) $f_{\mathbf{m}}(m)$. Furthermore, assume that the amplitude range of m is $-m_{\max} \leq m \leq m_{\max}$, that the uniform quantizer is of midrise type, and that the number of quantization levels is L. Then the quantization step-size is given by,

$$\Delta = \frac{2m_{\max}}{L} \tag{4.20}$$

Let $q = m - \hat{m}$ be the error introduced by the quantizer, then $-\Delta/2 \leq q \leq \Delta/2$. If the step-size Δ is sufficiently small (i.e., the number of quantization intervals L is sufficiently large), then it is reasonable to assume that the quantization error q is a *uniform* random variable over the range $[-\Delta/2, \Delta/2]$. The probability density function (pdf) of the random variable $\mathbf{q}$ is therefore given by,

$$f_{\mathbf{q}}(q) = \begin{cases} \frac{1}{\Delta}, & -\frac{\Delta}{2} < q \leq \frac{\Delta}{2} \\ 0, & \text{otherwise} \end{cases} \tag{4.21}$$

Note that with this assumption, the mean of the quantization error is zero, while its variance can be calculated as follows:

$$\begin{aligned} \sigma_q^2 &= \int_{-\Delta/2}^{\Delta/2} q^2 f_{\mathbf{q}}(q)\mathrm{d}q = \int_{-\Delta/2}^{\Delta/2} q^2 \left(\frac{1}{\Delta}\right) \mathrm{d}q \\ &= \frac{\Delta^2}{12} = \frac{m_{\max}^2}{3L^2} \end{aligned} \tag{4.22}$$

where the last equality follows from (4.20).

Each target level at the output of a quantizer is typically encoded (or represented) in binary form, i.e., a binary string. For convenience, the number of quantization levels is usually chosen to be a power of two, i.e., $L = 2^R$, where R is the number of binary bits needed to represent each target level. Substituting $L = 2^R$ into (4.22) one obtains the following expression for the variance of the quantization error:

$$\sigma_q^2 = \frac{m_{\max}^2}{3 \times 2^{2R}} \tag{4.23}$$

Since the message sample m is a zero-mean random variable whose pdf is $f_{\mathbf{m}}(m)$, the average power of the message is equal to the variance of m, i.e., $\sigma_m^2 = \int_{-m_{\max}}^{m_{\max}} m^2 f_{\mathbf{m}}(m)\mathrm{d}m$. Therefore, the signal-to-quantization noise ratio can be expressed as,

$$\mathrm{SNR_q} = \left(\frac{3\sigma_m^2}{m_{\max}^2}\right) 2^{2R} \tag{4.24}$$

$$= \frac{3 \times 2^{2R}}{F^2} \tag{4.25}$$

The parameter F in (4.25) is called the *crest factor* of the message, defined as,

$$F = \frac{\text{Peak value of the signal}}{\text{RMS value of the signal}} = \frac{m_{\max}}{\sigma_m} \tag{4.26}$$

Equation (4.25) shows that the signal-to-noise ratio of an uniform quantizer increases *exponentially* with the number of bits per sample R and decreases with the square of the message's crest factor. The message's crest factor is an inherent property of the signal source, while R is a technical specification, i.e., under an engineer's control.

Expressed in decibels, the signal-to-noise ratio is given by,

$$\begin{aligned}10\log_{10}\mathrm{SNR_q} &= 6.02R + 10\log_{10}\left(\frac{\sigma_m^2}{m_{\max}^2}\right) + 4.77 &(4.27)\\ &= 6.02R - 20\log_{10}F + 4.77 &(4.28)\end{aligned}$$

The above equation, called the *6-dB rule*, points out the significant performance characteristic of an uniform quantizer: *an additional 6-dB improvement in SNR is obtained for each bit added to represent the continuous signal sample.*

4.3.2 Optimal Quantizer

In the previous section the uniform quantizer was discussed where all the quantization regions are of equal size and the target (quantized) levels are at the midpoint of the quantization regions. Though simple, uniform quantizers are not optimal in terms of minimizing the signal-to-quantization noise ratio. In this section the optimal quantizer that maximizes the $\mathrm{SNR_q}$ is studied.

Consider a message signal $m(t)$ drawn from some stationary process. Let $[-m_{\max}, m_{\max}]$ be the amplitude range of the message, which is partitioned into L quantization regions as in (4.18). Instead of being equally spaced, the decision levels are constrained to satisfy only the following three conditions:

$$\begin{aligned}&D_1 = -m_{\max}\\ &D_{L+1} = m_{\max}\\ &D_l \le D_{l+1}, \quad \text{for } l = 1, 2, \ldots L\end{aligned} \qquad (4.29)$$

A target level may lie anywhere within its quantization region and as before, is denoted by T_k, $k = 1, \ldots, L$. Then the average quantization noise power is given by,

$$N_q = \sum_{l=1}^{L} \int_{D_l}^{D_{l+1}} (m - T_l)^2 f_{\mathbf{m}}(m)\mathrm{d}m \qquad (4.30)$$

The problem is to find the set of $2L - 1$ variables $\{D_2, D_3, \ldots, D_L, T_1, T_2, \ldots, T_L\}$ to maximize the signal-to-quantization noise power ratio, or equivalently to minimize the average power of the quantization noise N_q. Differentiating N_q with respect to a specific threshold, say D_j (use Leibniz's rule)† and setting the result to 0 yields

$$\frac{\partial N_{\mathrm{q}}}{\partial D_j} = f_{\mathbf{m}}(D_j)\left[(D_j - T_{j-1})^2 - (D_j - T_j)^2\right] = 0, \quad j = 2, 3, \ldots L. \tag{4.31}$$

The above gives $L-1$ equations with solutions:

$$D_l^{\mathrm{opt}} = \frac{T_{l-1} + T_l}{2}, \quad l = 2, 3, \ldots L. \tag{4.32}$$

This result simply means that, in an optimal quantizer, the decision levels are the midpoints of the target values (note that the target values of the optimal quantizer are not yet known).

To determine the L target values T_l, differentiate N_{q} with respect to a specific target level T_j and set the result to zero:

$$\frac{\partial N_{\mathrm{q}}}{\partial T_j} = -2\int_{D_j}^{D_{j+1}} (m - T_j) f_{\mathbf{m}}(m)\mathrm{d}m = 0, \quad j = 1, 2, \ldots L. \tag{4.33}$$

The above equation gives

$$T_l^{\mathrm{opt}} = \frac{\int_{D_l}^{D_{l+1}} m f_{\mathbf{m}}(m)\mathrm{d}m}{\int_{D_l}^{D_{l+1}} f_{\mathbf{m}}(m)\mathrm{d}m}, \quad l = 1, 2, \ldots, L \tag{4.34}$$

Equation (4.34) states that in an optimal quantizer the target value for a quantization region should be chosen to be the *centroid* (conditional expected value) of that region.

In summary, Equations (4.32) and (4.34) give the necessary and sufficient conditions for a memoryless quantizer to be optimal and are known as the *Lloyd-Max conditions*. Although the conditions are very simple, except in some exceptional cases, an analytical solution to the optimal quantizer design is not possible. Instead, the optimal quantizer is designed in an iterative manner as follows: start by specifying an arbitrary set of decision levels (for example the set that results in equal-length regions) and then find the target values using (4.34). Then determine the new decision levels using (4.32). The two steps are iterated until the parameters do not change significantly from one step to the next.

Though optimal, a major disadvantage of the optimal quantizer is that it requires knowledge of the statistical properties of the message source, namely the pdf $f_{\mathbf{m}}(m)$ of the message amplitude. In practice the quantizer in use may have to deal with a variety of sources. Another

† Leibniz's rule states that if $f(p) = \int_{l(p)}^{u(p)} g(x;p)\mathrm{d}x$, where p is a parameter, then $\frac{\partial f(p)}{\partial p} = \int_{l(p)}^{u(p)} \frac{\partial g(x;p)}{\partial p}\mathrm{d}x + g(x=u(p);p)\frac{\partial u(p)}{\partial p} - g(x=l(p);p)\frac{\partial l(p)}{\partial p}$.

disadvantage is that the quantizer is designed for a specific $m_{\max}$, while typically the signal level varies, resulting in poor performance. These disadvantages prevent the use of optimal quantizer in practical applications. In the next section, a different quantization method, used in practice, that overcomes these disadvantages is examined. The method is quite robust to both the source statistics and changes in the signal's power level.

4.3.3 Robust Quantizers

It can be easily verified from Equations (4.32) and (4.34) that for the special case where the message signal is uniformly distributed, the optimal quantizer is a uniform quantizer. Thus, as long as the distribution of the message signal is close to uniform, the uniform quantizer works fine. However, for certain signals such as voice, the input distribution is far from uniform. For a voice signal in particular, there exists a higher probability for smaller amplitudes (corresponding to silent periods and soft speech) and a lower probability for larger amplitudes (corresponding to loud speech). Therefore it is more efficient to design a quantizer with more quantization regions at lower amplitudes and less quantization regions at larger amplitudes to overcome the variations in power levels that the quantizer sees at its input. The resulting quantizer would be, in essence, a *nonuniform* quantizer having quantization regions of various sizes.

The usual and robust method for performing nonuniform quantization is to first pass the continuous samples through a monotonic nonlinearity called *compressor*, that compresses the large amplitudes (which essentially reduces the dynamic range of the signal). One view of the compressor is that it acts like a variable-gain amplifier; it has high gain at low amplitudes and less gain at high amplitudes. The compressed signal is applied to a uniform quantizer. At the receiving end, the inverse operation of the compressor is carried out by the *expander* to obtain the sampled values. The combination of a *com*pressor and an ex*pander* is called a *compander*. Figure 4.11 shows the block diagram of this technique, where $g(m)$ and $g^{-1}(m)$ are the compressing and expanding functions, respectively.

There are two types of companders that are widely used for voice signal in practical telecommunications systems. The μ-law compander used in the United States, Canada and Japan employs the following

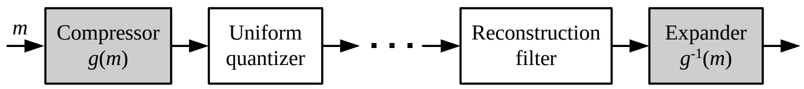


Fig. 4.11. Block diagram of compander technique.

logarithmic compressing function:

$$y = y_{\max} \frac{\ln\left[1 + \mu\left(|m|/m_{\max}\right)\right]}{\ln(1+\mu)} \operatorname{sgn}(m) \tag{4.35}$$

where

$$\operatorname{sgn}(x) = \begin{cases} +1, & \text{for } x \geq 0 \\ -1, & \text{for } x < 0 \end{cases} \tag{4.36}$$

In (4.35) $m_{\max}$ and $y_{\max}$ are the maximum positive levels of the input and output voltages respectively. The compression characteristic is shown in Figure 4.12-(a) for several values of μ. Note that $\mu = 0$ corresponds to a linear amplification, i.e., there is no compression. In the United States and Canada, the parameter μ was originally set to 100 for use with a 7-bit PCM encoder. It was later changed to 255 for use with an 8-bit encoder.

Another compression characteristic, used mainly in Europe, Africa and the rest of Asia, is the A-law, defined as

$$y = \begin{cases} y_{\max} \dfrac{A\left(|m|/m_{\max}\right)}{1 + \ln A} \operatorname{sgn}(m), & 0 < \dfrac{|m|}{m_{\max}} \leq \dfrac{1}{A} \\[2ex] y_{\max} \dfrac{1 + \ln\left[A\left(|m|/m_{\max}\right)\right]}{1 + \ln A} \operatorname{sgn}(m), & \dfrac{1}{A} < \dfrac{|m|}{m_{\max}} < 1 \end{cases} \tag{4.37}$$

where A is a positive constant. The A-law compression characteristic is shown in Figure 4.12-(b) for several values of A. Note that a standard value for A is 87.6.

4.3.4 $\mathrm{SNR_q}$ *of Non-Uniform Quantizers*

Consider a general compression characteristic as shown in Figure 4.13, where $g(m)$ maps the interval $[-m_{\max}, m_{\max}]$ into the interval $[-y_{\max}, y_{\max}]$. Note that the output of the compressor is uniformly quantized. Let y_l and Δ denote, respectively, the target level and the (equal) step-size of the lth quantization region for the compressed signal y. Recall that for an L-level midrise quantizer one has $\Delta = 2y_{\max}/L$. The corresponding

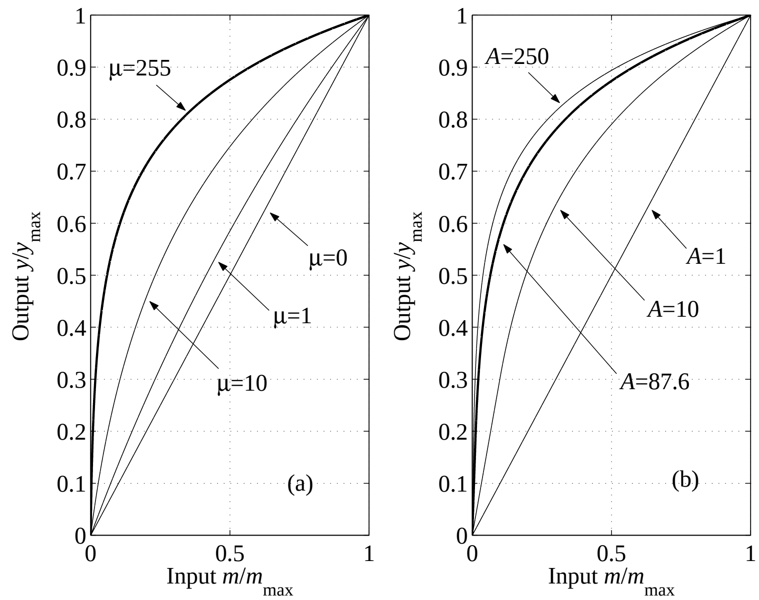


Fig. 4.12. Compression characteristic: (a) μ-law and (b) A-law. Note that both characteristics are *odd* functions and only the positive half of the input signal is illustrated.

target level and step-size of the lth region for the original signal m are m_l and Δ_l respectively.

Assume that the number of quantization intervals L is very large ($L \gg 1$) and the density function of the message m is smooth enough. Then both Δ and Δ_l are small. Thus one can approximate $f_{\mathbf{m}}(m)$ to be a constant $f_{\mathbf{m}}(m_l)$ over Δ_l and, as a consequence, the target level m_l is basically at the midpoint of the lth quantization region. With this approximation, N_{q} can be evaluated as follows:

$$\begin{aligned} N_{\mathrm{q}} &= \sum_{l=1}^{L} \int_{m_l-\frac{\Delta_l}{2}}^{m_l+\frac{\Delta_l}{2}} (m-m_l)^2 f_{\mathbf{m}}(m)\mathrm{d}m \\ &\cong \sum_{l=1}^{L} f_{\mathbf{m}}(m_l) \int_{m_l-\frac{\Delta_l}{2}}^{m_l+\frac{\Delta_l}{2}} (m-m_l)^2 \mathrm{d}m \\ &= \sum_{l=1}^{L} \frac{\Delta_l^3}{12} f_{\mathbf{m}}(m_l) \end{aligned} \quad (4.38)$$

Furthermore, it can be seen from Figure 4.13 that Δ is related to Δ_l

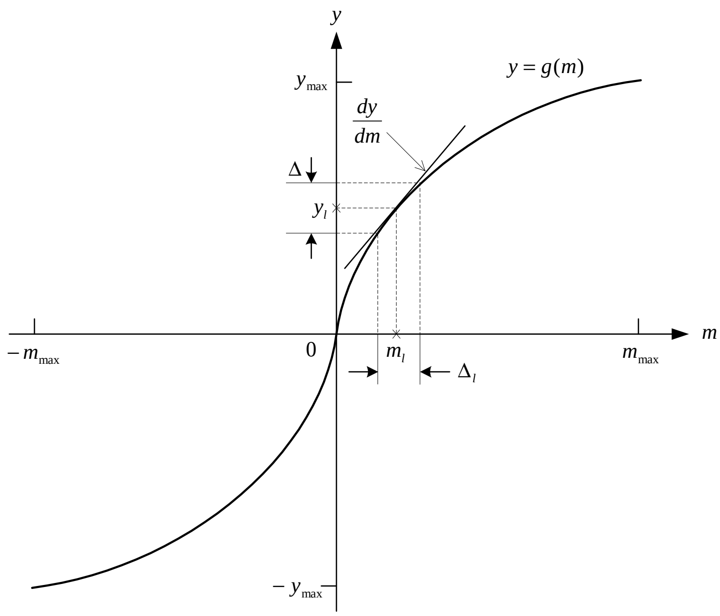


Fig. 4.13. Compression characteristic $g(m)$.

through the slope of $g(m)$ as follows:

$$\frac{\Delta}{\Delta_l} = \left.\frac{\mathrm{d}g(m)}{\mathrm{d}m}\right|_{m=m_l} \tag{4.39}$$

Substituting Δ_l found from (4.39) into (4.38) produces

$$N_{\mathrm{q}} = \frac{\Delta^2}{12}\sum_{l=1}^{L}\frac{f_{\mathbf{m}}(m_l)}{\left(\left.\frac{\mathrm{d}g(m)}{\mathrm{d}m}\right|_{m=m_l}\right)^2}\Delta_l \tag{4.40}$$

Finally, since $L \gg 1$ one can approximate the summation by an integral to obtain

$$\begin{aligned} N_{\mathrm{q}} &= \frac{\Delta^2}{12}\int_{-m_{\max}}^{m_{\max}}\frac{f_{\mathbf{m}}(m)}{\left(\frac{\mathrm{d}g(m)}{\mathrm{d}m}\right)^2}\mathrm{d}m \\ &= \frac{y_{\max}^2}{3L^2}\int_{-m_{\max}}^{m_{\max}}\frac{f_{\mathbf{m}}(m)}{\left(\frac{\mathrm{d}g(m)}{\mathrm{d}m}\right)^2}\mathrm{d}m \end{aligned} \tag{4.41}$$

As an example, let's determine $\mathrm{SNR_q}$ for the μ-law compander. To this end, evaluate the derivative of $g(m)$ in (4.35) to get

$$\frac{\mathrm{d}g(m)}{\mathrm{d}m} = \frac{y_{\max}}{\ln(1+\mu)} \frac{\mu(1/m_{\max})}{1+\mu(|m|/m_{\max})} \tag{4.42}$$

Note that the derivative of $g(m)$ is an even function. This is expected from the symmetry of the μ-law compression characteristic. Now substituting (4.42) into (4.41) gives

$$\begin{aligned} N_{\mathrm{q}} &= \frac{y_{\max}^2}{3L^2} \frac{\ln^2(1+\mu)}{y_{\max}^2\left(\frac{\mu}{m_{\max}}\right)^2} \int_{-m_{\max}}^{m_{\max}} \left[1+\mu\left(\frac{|m|}{m_{\max}}\right)\right]^2 f_{\mathbf{m}}(m)\mathrm{d}m \\ &= \frac{m_{\max}^2}{3L^2} \frac{\ln^2(1+\mu)}{\mu^2} \times \\ &\quad \int_{-m_{\max}}^{m_{\max}} \left[1+2\mu\left(\frac{|m|}{m_{\max}}\right)+\mu^2\left(\frac{|m|}{m_{\max}}\right)^2\right] f_{\mathbf{m}}(m)\mathrm{d}m \end{aligned} \tag{4.43}$$

Since $\int_{-m_{\max}}^{m_{\max}} f_{\mathbf{m}}(m)\mathrm{d}m = 1$, $\int_{-m_{\max}}^{m_{\max}} m^2 f_{\mathbf{m}}(m)\mathrm{d}m = \sigma_m^2$ and $\int_{-m_{\max}}^{m_{\max}} |m| f_{\mathbf{m}}(m)\mathrm{d}m = \mathcal{E}\{|m|\}$, the quantization noise power can be rewritten as,

$$N_{\mathrm{q}} = \frac{m_{\max}^2}{3L^2} \frac{\ln^2(1+\mu)}{\mu^2} \left[1+2\mu\frac{\mathcal{E}\{|m|\}}{m_{\max}}+\mu^2\frac{\sigma_m^2}{m_{\max}^2}\right] \tag{4.44}$$

Finally, by noting that the average power of message signal is σ_m^2, the signal-to-quantization noise power of the μ-law compander is given by,

$$\mathrm{SNR_q} = \frac{\sigma_m^2}{N_{\mathrm{q}}} = \frac{3L^2\mu^2}{\ln^2(1+\mu)} \frac{(\sigma_m^2/m_{\max}^2)}{1+2\mu(\mathcal{E}\{|m|\}/m_{\max})+\mu^2(\sigma_m^2/m_{\max}^2)} \tag{4.45}$$

To express $\mathrm{SNR_q}$ as a function of the normalized power level $\sigma_n^2 \cong \frac{\sigma_m^2}{m_{\max}^2}$, rewrite the term $\frac{\mathcal{E}\{|m|\}}{m_{\max}}$ in the denominator as $\frac{\mathcal{E}\{|m|\}}{\sigma_m}\frac{\sigma_m}{m_{\max}} = \frac{\mathcal{E}\{|m|\}}{\sigma_m}\sigma_n$. Therefore,

$$\mathrm{SNR_q}(\sigma_n^2) = \frac{3L^2}{\ln^2(1+\mu)} \frac{\sigma_n^2}{1+2\mu\sigma_n\frac{\mathcal{E}\{|m|\}}{\sigma_m}+\mu^2\sigma_n^2} \tag{4.46}$$

Equation (4.46) shows that the $\mathrm{SNR_q}$ of the μ-law compander depends on the statistics of the message through $\mathcal{E}\{|m|\}/\sigma_m$. For example, for a message with Gaussian density $\mathcal{E}\{|m|\}/\sigma_m = \sqrt{2/\pi} = 0.798$, and $\mathcal{E}\{|m|\}/\sigma_m = 1/\sqrt{2} = 0.707$ for a Laplacian-distributed message.

Furthermore, if $\mu \gg 1$ then the dependence of SNR_q on the message's characteristics is very small and SNR_q can be approximated as

$$\text{SNR}_\text{q} = \frac{3L^2}{\ln^2(1+\mu)} \tag{4.47}$$

For practical values of $\mu = 255$ and $L = 256$, one has $\text{SNR}_\text{q} = 38.1\text{dB}$.

To compare with the uniform quantizer, Figure 4.14 plots the SNR_q of the uniform quantizer and the μ-law quantizer over the same range of normalized input power. As can be seen from this figure, the μ-law quantizer can maintain a fairly constant SNR_q over a wide range of input power levels (and also for different input probability density functions). In contrast, the SNR_q of the uniform quantizer decreases linearly as the input power level drops. Note that at $\sigma_m^2/m_{\text{max}}^2 = 0\text{dB}$, the SNR_q for the μ-law and uniform quantizers are 38.1dB and 52.9dB respectively. Thus with the μ-law quantizer, one sacrifices performance for larger input power levels to obtain a performance that remains robust over a wide range of input levels.

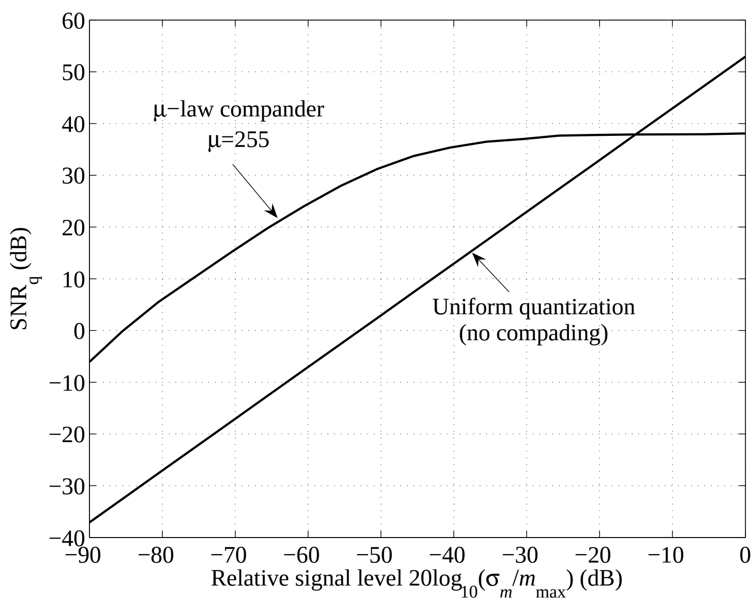


Fig. 4.14. Signal to quantization noise ratio of 8-bit quantizer with and without companding.

4.3.5 Differential Quantizers

In the quantizers looked at thus far each signal sample is quantized independently of all the others. However, most message signals (such as voice or video signals) sampled at the Nyquist rate or faster exhibit a high degree of correlation between successive samples. This essentially means that the signal does not change rapidly from one sample to the next. This redundancy can be exploited to obtain a better $\mathrm{SNR_q}$ for a given number of levels, L or conversely for a specified $\mathrm{SNR_q}$ the number of levels, L can be reduced. The number of bits, R, needed to represent L levels is $R = \log_2 L$. Reducing L means that the number of bits needed to represent a sample is reduced and hence the bit rate is reduced. A lower bit rate eventually implies that less bandwidth is needed by the communication system.

To motivate further the discussion two examples are considered: one a toy example which illustrates how redundancy helps in reducing bit rate, the other leads one into the major topic of this section, *differential quantizers*. First the toy example. Consider a message source whose output is the random process $\mathbf{m}(t) = \mathbf{A}\mathrm{e}^{-\mathbf{c}t}u(t)$, where $\mathbf{A}$ and $\mathbf{c}$ are random variables with arbitrary pdfs. In any transmission the output of the message source is a specific member of the ensemble, i.e., the signal $m(t) = A\mathrm{e}^{ct}u(t)$ where A and c are now specific, but unknown, values. One could of course sample this signal at a chosen rate T_s, quantize these samples and transmit the resultant sequence of samples. However after some thought one could equally just quantize two samples say at $t = 0$ and $t = t_1 > 0$ and transmit the quantized values $m(0) = A$, $m(t_1) = A\mathrm{e}^{ct_1}$. At the receiver the two sample values are used to determine the values of A and c which are then used to reconstruct the entire waveform.

Of course the above is an extreme example of redundancy in the message waveform. Indeed once one has the first two samples then one can predict exactly the sample values at all other sampling times. In the more practical case one does not have nice analytical expressions for how the waveform is generated to enable this perfect prediction. The typical information one has about the redundancy in the samples, as mentioned above, is the correlation between them. The approach, as shown in Figure 4.15 where $m[n]$ is used instead of $m(nT_s)$ to refer to the sampled values, is to use the previous sample values to predict the next sample value and then transmit the difference. Recall from (4.23) that the quantization noise, σ_q^2, depends directly on the message range, i.e., $m_{\max}$. The message being quantized and transmitted now is the prediction error, $e[n] = m[n] - \tilde{m}[n] = m[n] - km[n]$. Therefore if $|e_{\max}| = |m_{\max} - km_{\max}| = |1-k|m_{\max}$ is less than $m_{\max}$ then the quantization noise power is reduced. Therefore the predictor should be such that $0 < k < 2$, ideally $k = 1$, hopefully $k \approx 1$. The approach just described is called differential quantization. The main design issue for this quantizer is how to predict the message sample value. Ideally one would base the design on a criterion that minimizes $e_{\max}$ but this turns out to be intractable. Thus the predictor design is based on the minimization of the error variance, σ_e^2. Further a linear structure is imposed on the predictor since it is both straightforward to design, quite practical to realize, and performs well.

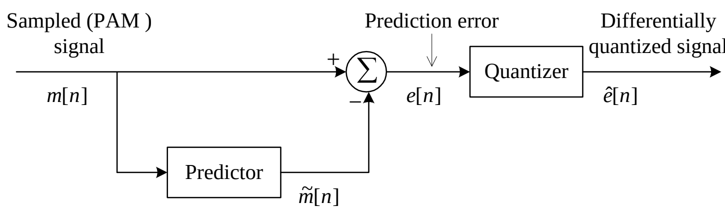


Fig. 4.15. Illustration of a differential quantizer.

Linear Predictor. A linear predictor predicts the current sample based on a weighted sum of the previous p samples. It is called a pth-order linear predictor. As can be seen in Figure 4.16, the predicted sample, $\tilde{m}[n]$, is given by†

$$\tilde{m}[n] = \sum_{i=1}^{p} w_i m[n-i] \tag{4.48}$$

The blocks labeled with z^{-1} in Figure 4.16 signify unit-delay elements (z-transform notation).

In general, the coefficients $\{w_i\}$ are selected to *minimize* some function of the error between $m[n]$ and $\tilde{m}[n]$. However as mentioned it is the average power of the prediction error that is of interest. The coefficient

† Note that for simplicity of presentation, the notations used here slightly different from those in Figure 4.15, where the input of the predictor is $\hat{m}[n]$ rather than $m[n]$.

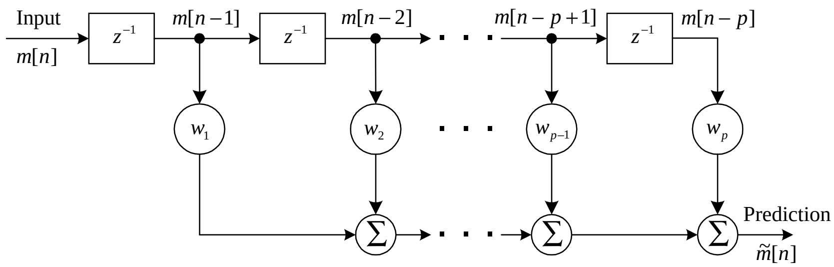


Fig. 4.16. Block diagram of a pth-order linear predictor.

set $\{w_i\}$ is selected to minimize

$$\begin{aligned}\sigma_e^2 = \mathcal{E}\{e^2[n]\} &= \mathcal{E}\left\{\left(m[n] - \sum_{i=1}^{p} w_i m[n-i]\right)^2\right\} \\ &= \mathcal{E}\{m^2[n]\} - 2\sum_{i=1}^{p} w_i \mathcal{E}\{m[n]m[n-i]\} \\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \mathcal{E}\{m[n-i]m[n-j]\} \end{aligned} \tag{4.49}$$

Since the quantity $\mathcal{E}\{m[n]m[n+k]\}$ is the *autocorrelation* function, $R_{\mathbf{m}}(k)$ of the sampled signal sequence $\{m[n]\}$, (4.49) can be expressed as follows:

$$\sigma_e^2 = R_{\mathbf{m}}(0) - 2\sum_{i=1}^{p} w_i R_{\mathbf{m}}(i) + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j R_{\mathbf{m}}(i-j) \tag{4.50}$$

Now take the partial derivative of σ_e^2 with respect to each coefficient w_i and set the results to zero to yield a set of linear equations:

$$\sum_{i=1}^{p} w_i R_{\mathbf{m}}(i-j) = R_{\mathbf{m}}(j), \quad j = 1, 2, \ldots, p \tag{4.51}$$

The above collection of equations can be arranged in matrix form as shown in (4.52). This set of equation is also known as the *normal equa-*

tions or the *Yule-Walker equations*.

$$\begin{bmatrix} R_{\mathbf{m}}(0) & R_{\mathbf{m}}(-1) & R_{\mathbf{m}}(-2) & \cdots & R_{\mathbf{m}}(-p+1) \\ R_{\mathbf{m}}(1) & R_{\mathbf{m}}(0) & R_{\mathbf{m}}(-1) & \cdots & R_{\mathbf{m}}(-p+2) \\ R_{\mathbf{m}}(2) & R_{\mathbf{m}}(1) & R_{\mathbf{m}}(0) & \cdots & R_{\mathbf{m}}(-p+3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{\mathbf{m}}(p-1) & R_{\mathbf{m}}(p-1) & R_{\mathbf{m}}(p-3) & \cdots & R_{\mathbf{m}}(0) \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_p \end{bmatrix} = \begin{bmatrix} R_{\mathbf{m}}(1) \\ R_{\mathbf{m}}(2) \\ R_{\mathbf{m}}(3) \\ \vdots \\ R_{\mathbf{m}}(p) \end{bmatrix} \tag{4.52}$$

The remaining design issue is the choice of p, the predictor's order. In practice for voice signals it has been found that greatest improvement happens when one goes from no prediction to a first-order prediction, i.e., $p = 1$. But for other sources one should be prepared to experiment.

Reconstruction of $m[n]$ from the Differential Samples. Though at the receiver one sees the quantized samples, $\hat{e}[n]$, for discussion purposes we assume that the quantization noise or error is negligible. The distortion introduced by the quantizer is ignored and therefore we look at the reconstruction of $m[n]$ from the differential samples $e[n]$. For the pth-order linear predictor the differential sample $e[n]$ is related to $m[n]$, using (4.48), by

$$e[n] = m[n] - \sum_{i=1}^{p} w_i m[n-i]. \tag{4.53}$$

Using the z-transform, the above relationship becomes

$$\begin{aligned} e(z^{-1}) &= m(z^{-1}) - \sum_{i=1}^{p} w_i z^{-i} m(z^{-1}) \\ &= m(z^{-1}) - m(z^{-1}) \sum_{i=1}^{p} z^{-i} \\ &= m(z^{-1}) - m(z^{-1}) H(z^{-1}) \end{aligned} \tag{4.54}$$

where $H(z^{-1}) = \sum_{i=1}^{p} w_i z^{-i}$ is the transfer function of the linear predictor.

Equation (4.54) states that $e(z^{-1}) = m(z^{-1})\left[1 - H(z^{-1})\right]$, or

$$m(z^{-1}) = \frac{1}{1 - H(z^{-1})} e(z^{-1}). \quad (4.55)$$

Recall from feedback control theory that the input/output relationship for negative feedback is $\frac{G}{1+GH}$, where G is the forward loop gain and H is the feedback loop gain. Comparing this with (4.55) one concludes that $G = 1$ and $H = -H(z^{-1})$. Therefore $m(z^{-1}$ is reconstructed by the block diagram of Figure 4.17. Note that even if one has forgotten or has never learned feedback control theory one should be able to convince oneself that the block diagram on the right in Figure 4.17 reconstructs $m[n]$.

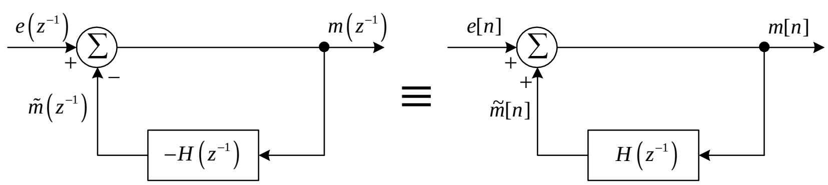


Fig. 4.17. Reconstruction of $m[n]$ from $e[n]$.

However the input to the reconstruction filter is not $e[n]$ but $\hat{e}[n]$, which is equal to $e[n]+q[n]$ where $q[n]$ is the quantization noise or error. The difficulty this poses with the proposed reconstruction filter is that not only does the noise in the present differential sample affect the reconstructed signal but also all the previous quantzation noise samples, though with diminishing effect. In essence this is because the reconstruction filter has infinite memory. To add to this effect one should also be aware that the channel will also induce noise in $e[n]$.

A solution that would eliminate this effect is quite desirable. Since engineers are supposed to be, if anything, ingenious this is precisely what is accomplished by the system shown in Figure 4.18, known as differential pulse code modulation (DPCM). The pulse code aspect is discussed in the next section but here we show that indeed $\hat{m}[n]$ is only affected by the quantization noise, $q[n]$. The analysis shows that this is true regardless of the predictor used, it can even be nonlinear, and the conclusion is also independent of the form of the quantizer. Truly ingenious.

Referring to Figure 4.15-(a), the input signal to the quantizer is given

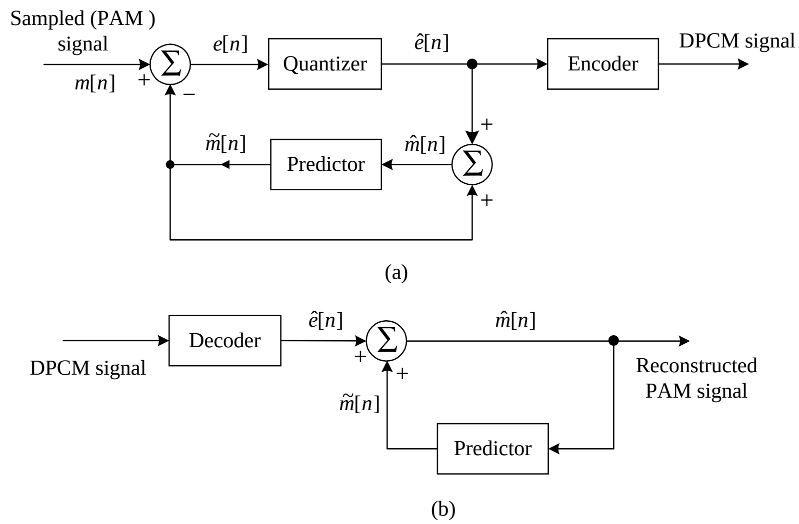


Fig. 4.18. DPCM system using a differential quantizer: (a) Transmitter and (b) Receiver.

by

$$e[n] = m[n] - \widetilde{m}[n] \tag{4.56}$$

which is the difference between the unquantized sample $m[n]$ and its prediction, denoted by $\widetilde{m}[n]$. The predicted value can be obtained using a linear prediction filter, whose input is $\hat{m}[n]$. The output of the quantizer is then encoded to produce the DPCM signal. The quantizer output can be expressed as,

$$\hat{e}[n] = e[n] - q[n] \tag{4.57}$$

where $q[n]$ is the quantization error. According to Figure 4.15-(a), the input to the predictor can be written as follows.

$$\begin{aligned} \hat{m}[n] &= \widetilde{m}[n] + \hat{e}[n] \\ &= \widetilde{m}[n] + (e[n] - q[n]) \\ &= (\widetilde{m}[n] + e[n]) - q[n] \\ &= m[n] - q[n] \end{aligned} \tag{4.58}$$

Since $q[n]$ is the quantization error, Equation (4.58) implies that $\hat{m}[n]$ is just the quantized version of the input sample $m[n]$.

The receiver for a DPCM system is shown in Figure 4.15-(b). In the absence of the channel noise, the output of the decoder is identical to the input of the encoder (at the transmitter), which is $\hat{e}[n]$. If the predictor at the receiver is the same as the one in the transmitter then its output should be equal to $\widetilde{m}[n]$. It follows that the final output of the receiver is $\hat{e}[n] + \widetilde{m}[n]$, which is equal to $\hat{m}[n]$. Thus, according to (4.58), the output of the receiver differs from the original input $m[n]$ only by the quantization error $q[n]$ incurred as a result of quantizing the prediction error. Note that the quantization errors do not accumulate in the receiver.

4.4 Pulse Code Modulation (PCM)

The last block in Figure 4.7 to be discussed is the *encoder*. A PCM signal is obtained from the quantized PAM signal by encoding each quantized sample to a *digital codeword*. If the PAM signals are quantized using L target levels, then in binary PCM each quantized sample is digitally encoded into an R-bit binary codeword, where $R = \lceil \log_2 L \rceil + 1$. The quantizing and encoding operations are usually performed in the same circuit known as an analog-to-digital converter. The advantage of having PCM signal over a quantized PAM signal is that the binary digits of a PCM signal can be transmitted using many efficient modulation schemes compared to the transmission of a PAM signal. The topics of baseband and passband modulation of binary digits are covered in later chapters.

There are several ways to establish a one-to-one correspondence between target levels and the codeword. A convenient method, known as natural binary coding (NBC), is to express the ordinal number of the target level as a binary number as described in Figure 4.19. Another popular mapping method is called *Gray* mapping. Gray mapping is important in the demodulation of the signal because the most likely errors caused by noise involve the erroneous selection of an adjacent target level to the transmitted target level. Still another mapping, called foldover binary coding (FBC), is also sometimes encountered.

4.5 Summary

The basic concepts behind converting an analog information to a digital one have been presented in this chapter. The first step in this process is to sample the continuous-time output of the analog source. The sampling theorem shows that if the sampling rate is at least twice the

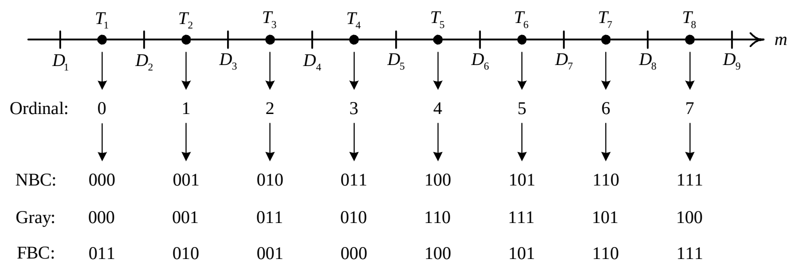


Fig. 4.19. Encoding of quantized levels into PCM codewords, $L = 8$.

highest frequency present in the analog signal then the original signal can be recovered exactly.

However not only does time axis need to be discretized, the analog amplitude values of the samples also must be quantized. Since this step is a many-to-one mapping, it inherently produces a distortion or a loss. Various quantizers to minimize this distortion, with a performance measure of signal-to-quantization noise power ratio, have been analyzed and discussed.

The final step, encoding, is to map quantized sample values to a set of discrete symbols, typically a sequence of binary digits called variously a binary string, binary word, or simply codeword. The codeword however is an abstract quantity, i.e., a nonphysical entity. To transmit the information that a codeword represents one needs to map each bit to an electrical signal or waveform. As has been mentioned several times this mapping is called modulation and it, along with demodulation, is the subject of succeeding chapters. To conclude we are finally in a position to discuss digital communications.

However a few further closing remarks are in order. The quantization and encoding discussed in this chapter go under the general topic of source coding. But some sources, as pointed out the keyboard, are already discrete in nature. The sampling/quantization steps are not necessary for them and only the encoding step that of converting the discrete source symbols to a codeword is required. Traditional source coding was concerned primarily with this step. Some aspects of this are presented in the problems at the end of the chapter but the interested reader is referred to the literature for a more comprehensive treatment.

Problems

4.1 (*PCM*) You have a disk with a capacity of approximately 600×10^6 bytes, and you wish to store a digital representation of the output of a source on the disk. Estimate the maximum recording duration for each of the following sources, using a straightforward PCM representation.

(a) 4 kHz speech, 8 bits per sample.
(b) 22 kHz audio signal, stereo, 16 bits per sample in each stereo channel.
(c) 5 MHz video signal, 12 bits per sample, combined with the audio signal above.
(d) Digital surveillance video, 1024×768 pixels per frame, 8 bits per pixel, 1 frame per second.

4.2 (*Crest Factor*) A signal parameter of interest in designing a quantizer is the crest factor F, defined as

$$F = \frac{\text{Peak value of the signal}}{\text{RMS value of the signal}} \tag{P4.1}$$

Determine the crest factor for the following signals:

(a) A sinusoid with peak amplitude of $m_{\max}$.
(b) A square wave of period T and amplitude range $[-m_{\max}, m_{\max}]$.

Above are deterministic signals and most of you are quite familiar with their crest factors. Now consider random signals whose probability density functions are:

(c) Uniform over the amplitude range $[-m_{\max}, m_{\max}]$.
(d) Zero-mean Gaussian with variance σ_m^2:
$f_{\mathbf{m}}(m) = \frac{1}{\sqrt{2\pi\sigma_m^2}} \exp\left(-\frac{m^2}{2\sigma_m^2}\right)$.
(e) Zero-mean Laplacian: $f_{\mathbf{m}}(m) = \frac{c}{2}\exp(-c|m|)$.
(f) Zero-mean Gamma: $f_{\mathbf{m}}(m) = \sqrt{\frac{k}{4\pi|m|}}\exp(-k|m|)$. Note that the Gamma distribution is sometimes used to model the voice signal.

Hint: For parts (d), (e) and (f), the peak value appears to be infinity. Thus take an "engineering approach" and define the peak value $m_{\max}$ such that the probability that the signal amplitude falls into $[-m_{\max}, m_{\max}]$ is 99%. Furthermore, evaluation of $m_{\max}$ for (d) and (f) involves the *error function*, defined as:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2} \mathrm{d}t \tag{P4.2}$$

The error function is available in MATLAB under the name `erf` (type `help erf` in MATLAB to see).

4.3 (*A different definition for* $\mathrm{SNR_q}$) For an analog signal with peak magnitude $m_{\max}$, let us define the *peak signal-to-quantization noise ratio* as

$$\mathrm{PSNR_q} = 20\log_{10}\left(\frac{m_{\max}}{q_{\max}}\right) \tag{P4.3}$$

where $q_{\max} = \max\{|m - \hat{m}|\}$ is the peak magnitude of the quantization noise. Assuming an R-bit uniform quantizer, what is the smallest value of R that ensures the $\mathrm{PSNR_q}$ is at least D dB?

4.4 (*Optimal quantizer*) Consider a message source that with confidence you feel is well modelled by a first order Laplacian density function, i.e., $f_{\mathbf{m}}(m) = \frac{c}{2}\exp(-c|m|)$. You wish to design an L-level optimum quantizer for the message. The equations that need to satisfied are:

$$D_l = \frac{T_{l-1} + T_l}{2}, \quad l = 2, \ldots, L \tag{P4.4}$$

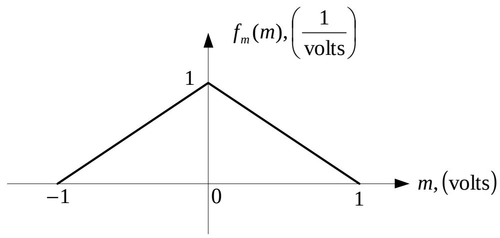
$$T_l = \frac{\int_{D_l}^{D_{l+1}} m f_{\mathbf{m}}(m)\mathrm{d}m}{\int_{D_l}^{D_{l+1}} f_{\mathbf{m}}(m)\mathrm{d}m}, \quad l = 1, \ldots, L \tag{P4.5}$$

(a) The form of Equation (P4.5) depends on $f_{\mathbf{m}}(m)$. Derive equation for T_l for the given message source.
Hint: Since $f_{\mathbf{m}}(m)$ is symmetric about zero, one needs to worry only about the positive m axis, i.e., there is a decision (threshold) level at $m = 0$ and you only need to determine half the target values. Thus when you determine T_l, do it for $m > 0$.

(b) What are the decision and target values for the following simple cases: (i) $L = 1$, (ii) $L = 2$ and (iii) $L = 4$.

4.5 Let $m(t)$ be a message that has the following amplitude probability density function:

(a) Design 1-bit ($L = 2$) quantizer that minimizes the average power of the quantization error.



(b) Compute the signal to quantization noise ratio SNR_q and put the answer in dB unit.

5

Optimum Receiver for Binary Data Transmission

In Chapter 4 it was shown how continuous waveforms are transformed into binary digits (or bits) via the processes of sampling, quantization and encoding (commonly referred to as pulse code modulation, or PCM). However, it should be pointed out that bits are just abstractions and there is nothing *physical* about the bits. Thus, for the transmission of the information, we need something physical to represent or "carry" the bits.

Here, we represent the binary digit b_k (0 or 1) by one of two electrical waveforms $s_1(t)$ and $s_2(t)$. Such a representation is the function of the *modulator* as shown in Figure 5.1. These waveforms are transmitted through the channel and possibly perturbed by the *noise*. At the receiving side, the *receiver* needs to make a decision on the transmitted bit $\hat{b}_k$ based on the received signal $r(t)$.

In this chapter we study the *optimum* receiver for a *binary* digital communication system as illustrated in Figure 5.1. The performance of

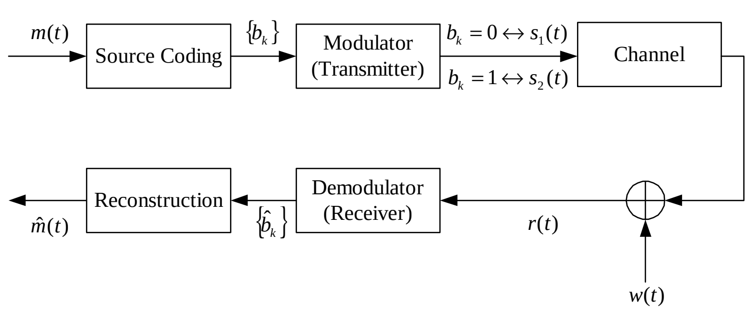


Fig. 5.1. Block diagram of a binary digital communication system.

the receiver will be measured in terms of error probability. A channel model of *infinite* bandwidth with additive white Gaussian noise is assumed. Though somewhat idealized, this channel model does provide us with a starting point to develop necessary approaches to designing and analyzing receivers. Besides, many communication channels such as satellite and deep space communications are well represented by this model.

Other assumptions and notations which pertain to the discussion of the communication system in Figure 5.1 are as follows.

- The bit duration of b_k is T_b seconds, or the bit rate is $r_b = 1/T_b$ (bits/sec).
- Bits in two different time slots are *statistically independent*.
- The *a priori* probability (i.e., before we observe the received signal $r(t)$) of b_k is given as follows:

$$\Pr[b_k = 0] = P_1, \tag{5.1}$$

$$\Pr[b_k = 1] = P_2, \tag{5.2}$$

  where $P_1 + P_2 = 1$. Most often, we will assume that the bits are equally likely, i.e., $P_1 = P_2 = 1/2$.
- The bit b_k is mapped by the modulator into one of the two signals $s_1(t)$ and $s_2(t)$. Each signal is of duration T_b seconds and has a finite energy:

$$E_1 = \int_0^{T_b} s_1^2(t)\mathrm{d}t \text{ (joules)}, \quad E_1 < \infty, \tag{5.3}$$

$$E_2 = \int_0^{T_b} s_2^2(t)\mathrm{d}t \text{ (joules)} \quad E_2 < \infty. \tag{5.4}$$

  For now we shall consider the signal waveforms to be *arbitrary*, but known to the receiver. Later in Chapters 6 and 7, by employing certain waveforms for $s_1(t)$ and $s_2(t)$, many popular baseband/passband modulation schemes are analyzed.
- The channel is wideband enough so that the signals $s_1(t)$ and $s_2(t)$ pass through without any distortion. Essentially, this means that there is no inter-symbol interference (ISI) between successive bits.
- The noise $w(t)$ is *stationary* Gaussian, zero-mean white noise with two-sided power spectral density of $N_0/2$ (watts/Hz). This implies that the pdf of the noise amplitude at any time instant t is Gaussian and

$$\mathcal{E}\{w(t)\} = 0, \quad \mathcal{E}\{w(t)w(t+\tau)\} = \frac{N_0}{2}\delta(\tau). \tag{5.5}$$

Given the above, the received signal over the time interval $[(k-1)T_b, kT_b]$, i.e., the interval that we send the bit b_k, can be written as,

$$r(t) = s_i(t-(k-1)T_b) + w(t), \quad (k-1)T_b \leq t \leq kT_b. \tag{5.6}$$

The objective is to design a receiver (or demodulator) such that by observing the signal $r(t)$, the probability of making an error is minimized. The development proceeds by reducing the problem from the observation of a time waveform to that of observing a set of numbers (which are random variables).†

The approach is basically to represent the signal $s_1(t)$, $s_2(t)$ and the noise, $w(t)$, by a specifically chosen series. The coefficients of this series then constitute the set of numbers on which our decision is based. This approach leads to a geometrical interpretation which readily lends insight and interpretation to the relationship between signal energy, distance between signals and error probability.

5.1 Geometric Representation of Signals $s_1(t)$ and $s_2(t)$

Consider two arbitrary signals $s_1(t)$ and $s_2(t)$. We wish to represent them as linear combinations of two *orthonormal* functions $\phi_1(t)$ and $\phi_2(t)$. Recall that two functions $\phi_1(t)$ and $\phi_2(t)$ are said to be orthornormal if the following two conditions are satisfied:

$$\int_0^{T_b} \phi_1(t)\phi_2(t)\mathrm{d}t = 0 \ (\textit{orthogonality}), \tag{5.7}$$

$$\int_0^{T_b} \phi_1^2(t)\mathrm{d}t = \int_0^{T_b} \phi_2^2(t)\mathrm{d}t = 1 \ (\text{normalized to have } \textit{unit energy}). \tag{5.8}$$

The signals $s_1(t)$ and $s_2(t)$ should have the following expansions in terms of $\phi_1(t)$ and $\phi_2(t)$:

$$s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t), \tag{5.9}$$

$$s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t). \tag{5.10}$$

By direct substitution and integral evaluation, it is simple to verify that if one can represent $s_1(t)$ and $s_2(t)$ as in (5.9) and (5.10), then the coefficients s_{ij}, $i,j \in \{1,2\}$ must be given as follows:

$$s_{ij} = \int_0^{T_b} s_i(t)\phi_j(t)\mathrm{d}t, \quad i,j \in \{1,2\}, \tag{5.11}$$

† A common name for this set is sufficient statistics, another is observables.

where we can view the mathematical operation $\int_0^{T_b} s_i(t)\phi_j(t)\mathrm{d}t$ as the projection of signal $s_i(t)$ onto the jth axis, $\phi_j(t)$. Equations (5.9) and (5.10) can be geometrically interpreted as shown in Figure 5.2.

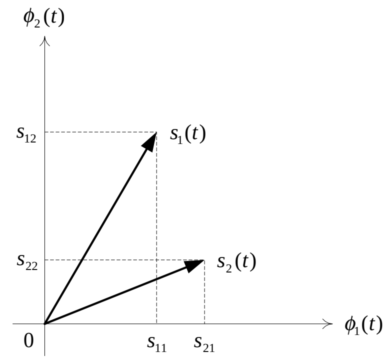


Fig. 5.2. Projection of $s_1(t)$ and $s_2(t)$ onto $\phi_1(t)$ and $\phi_2(t)$.

The question now is how do we choose the time functions $\phi_1(t)$ and $\phi_2(t)$, so that (5.7) is satisfied and also that $s_1(t)$ and $s_2(t)$ can be exactly represented by them. More fundamentally, we could ask if such a choice is possible. If possible, is the choice *unique*? As we shall see in the following, the choice is not only possible but there are many possible choices!

The most straightforward way to find $\phi_1(t)$ and $\phi_2(t)$ is as follows:

(i) Let $\phi_1(t) \equiv \dfrac{s_1(t)}{\sqrt{E_1}}$. This means that the first orthonormal function is chosen to be identical in shape to the first signal, but is normalized by the the square-root of the signal energy. Note that from Equation (5.9), one has $s_{11} = \sqrt{E_1}$ and $s_{12} = 0$.

(ii) To find $\phi_2(t)$, we first project $s_2^{'}(t) = \dfrac{s_2(t)}{\sqrt{E_2}}$ onto $\phi_1(t)$ and call this quantity ρ, the *correlation coefficient*:

$$\rho = \int_0^{T_b} \frac{s_2(t)}{\sqrt{E_2}}\phi_1(t)\mathrm{d}t = \frac{1}{\sqrt{E_1 E_2}}\int_0^{T_b} s_1(t)s_2(t)\mathrm{d}t. \quad (5.12)$$

(iii) Now subtract the projection $\rho\phi_1(t)$ from $s_2^{'}(t) = \dfrac{s_2(t)}{\sqrt{E_2}}$ to obtain

$$\phi_2^{'}(t) = \frac{s_2(t)}{\sqrt{E_2}} - \rho\phi_1(t). \quad (5.13)$$

Intuitively we would expect $\phi_2^{'}(t)$ to be orthogonal to $\phi_1(t)$. To verify that this is true, consider

$$\begin{aligned}\int_0^{T_b} \phi_2^{'}(t)\phi_1(t)\mathrm{d}t &= \int_0^{T_b} \frac{s_2(t)}{\sqrt{E_2}}\phi_1(t)\mathrm{d}t - \rho\int_0^{T_b} \phi_1^2(t)\mathrm{d}t \\ &= \rho - \rho = 0. \end{aligned} \tag{5.14}$$

(iv) The only thing left to do is to normalize $\phi_2^{'}(t)$ to make sure that $\phi_2(t)$ has unit energy (see Problem 5.1):

$$\begin{aligned}\phi_2(t) &= \frac{\phi_2^{'}(t)}{\sqrt{\int_0^{T_b}\left[\phi_2^{'}(t)\right]^2\mathrm{d}t}} = \frac{\phi_2^{'}(t)}{\sqrt{1-\rho^2}} \\ &= \frac{1}{\sqrt{1-\rho^2}}\left[\frac{s_2(t)}{\sqrt{E_2}} - \frac{\rho s_1(t)}{\sqrt{E_1}}\right]. \end{aligned} \tag{5.15}$$

To summarize, the two orthonormal functions are given by,

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}, \tag{5.16}$$

$$\phi_2(t) = \frac{1}{\sqrt{1-\rho^2}}\left[\frac{s_2(t)}{\sqrt{E_2}} - \frac{\rho s_1(t)}{\sqrt{E_1}}\right] \tag{5.17}$$

Furthermore, the coefficients s_{21} and s_{22} can be determined as follows:

$$s_{21} = \int_0^{T_b} s_2(t)\phi_1(t)\mathrm{d}t = \rho\sqrt{E_2}, \tag{5.18}$$

$$\begin{aligned}s_{22} &= \int_0^{T_b} s_2(t)\phi_2(t)\mathrm{d}t = \int_0^{T_b} s_2(t)\frac{1}{\sqrt{1-\rho^2}}\left[\frac{s_2(t)}{\sqrt{E_2}} - \frac{\rho s_1(t)}{\sqrt{E_1}}\right]\mathrm{d}t \\ &= \frac{1}{\sqrt{1-\rho^2}}\left(\sqrt{E_2} - \rho^2\sqrt{E_2}\right) = \left(\sqrt{1-\rho^2}\right)\sqrt{E_2}. \end{aligned} \tag{5.19}$$

Geometrically, the above procedure is illustrated in Figure 5.3.

To carry the geometric ideas further, define the distance between the signals $s_2(t)$ and $s_1(t)$ as,

$$d_{21} = \sqrt{\int_0^{T_b} [s_2(t) - s_1(t)]^2\mathrm{d}t}. \tag{5.20}$$

From the geometry of the signal representation we see that

$$\begin{aligned}d_{21}^2 &= s_{22}^2 + \left(\sqrt{E_1} - s_{21}\right)^2 \\ &= \left(1-\rho^2\right)E_2 + \left(E_1 - 2\rho\sqrt{E_1E_2} + \rho^2E_2\right) \\ &= E_1 - 2\rho\sqrt{E_1E_2} + E_2. \end{aligned} \tag{5.21}$$

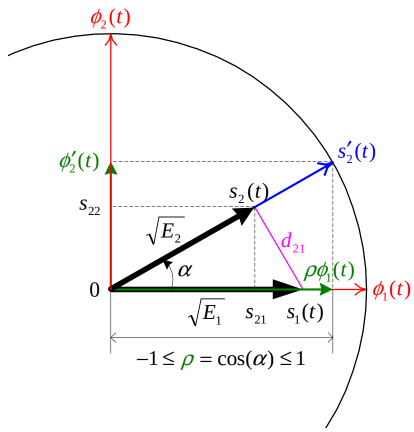


Fig. 5.3. Obtaining $\phi_1(t)$ and $\phi_2(t)$ from $s_1(t)$ and $s_2(t)$.

It can be verified that (try it yourself) evaluating (5.20) yields the same result of (5.21) for the distance d_{21}.

An important geometric interpretation follows directly from (5.21). In particular, by setting one of the two signals to zero, the distance between the two signals becomes the distance from the nonzero signal to the origin. This distance is simply the square root of the energy of the nonzero signal. For example, setting $s_1(t) = 0$ makes $E_1 = 0$ in (5.21) and results in $d_{21}^2 = E_2$, i.e., $d_{21} = \sqrt{E_2}$.

The above procedure has shown that, two arbitrary deterministic waveforms can be represented *exactly* by at most two properly chosen orthonormal functions. This procedure can be generalized to a set with an arbitrary number of finite-energy signals (which are not necessarily limited to $[0, T_b]$ as in the case of two signals just considered) and it is known as the *Gram-Schmidt* orthogonalization procedure. The Gram-Schmidt procedure is outlined below for the set of M waveforms.

Gram-Schmidt Procedure. Suppose that we have a set of M finite-energy waveforms $\{s_i(t), i = 1, 2 \ldots, M\}$ and we wish to construct a set of orthonormal waveforms that can represent $\{s_i(t), i = 1, 2 \ldots, M\}$ exactly. The first orthonormal function is simply constructed as,

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{\int_{-\infty}^{\infty} s_1^2(t)\mathrm{d}t}}. \tag{5.22}$$

The subsequent orthonormal functions are found as follows:

$$\phi_i(t) = \frac{\phi_i^{'}(t)}{\sqrt{\int_{-\infty}^{\infty}\left[\phi_i^{'}(t)\right]^2 \mathrm{d}t}}, \quad i = 2, 3, \ldots, N, \tag{5.23}$$

where

$$\phi_i^{'}(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}\phi_j(t), \tag{5.24}$$

$$s_{ij} = \int_{-\infty}^{\infty} s_i(t)\phi_j(t)\mathrm{d}t, \quad j = 1, 2, \ldots, i-1. \tag{5.25}$$

In general, the number of orthonormal functions, N, is less than or equal to the number of given waveforms, M, depending on one of the two possibilities:

(i) If the waveforms $\{s_i(t), i = 1, 2 \ldots, M\}$ form a *linearly independent set*, then $N = M$.

(ii) If the waveforms $\{s_i(t), i = 1, 2 \ldots, M\}$ are not linearly independent, then $N < M$.

To illustrate the Gram-Schmidt procedure, let us consider a few examples.

Example 5.1. Consider a signal set shown in Figure 5.4-(a). This signal set is in a sense a degenerate case since only one basis function $\phi_1(t)$ is needed to represent it. The signal energies are given by

$$E_1 = \int_0^{T_b} s_1^2(t)\mathrm{d}t = V^2 T_b = E_2 \equiv E \text{ (joules)}. \tag{5.26}$$

The first orthonormal function is given by

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E}} = \frac{s_1(t)}{\sqrt{V^2 T_b}}. \tag{5.27}$$

The correlation coefficient, ρ, is $\rho = \int_0^{T_b} \frac{s_2(t)s_1(t)}{E}\mathrm{d}t = -1$, reflecting the fact that $s_2(t) = -s_1(t)$. Therefore the unnormalized basis function $\phi_2^{'}(t)$ is

$$\phi_2^{'}(t) = \frac{s_2(t)}{\sqrt{E}} - \rho\phi_1(t) = \frac{s_2(t) + s_1(t)}{\sqrt{E}} = 0. \tag{5.28}$$

The basis function $\phi_1(t)$ is plotted in Figure 5.4-(b). In terms of $\phi_1(t)$

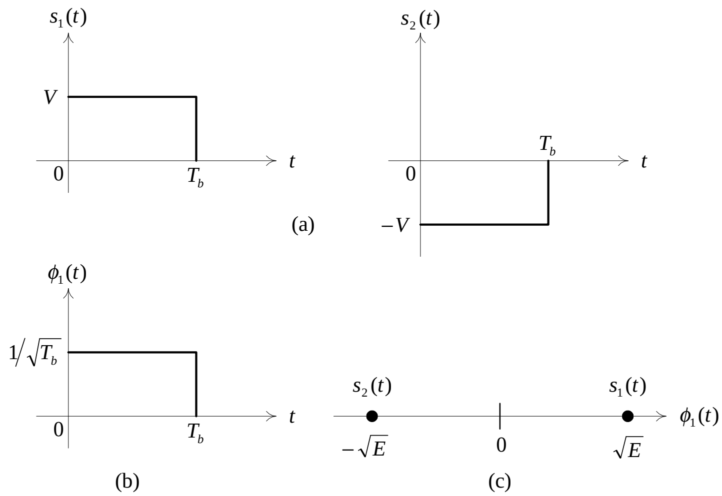


Fig. 5.4. Example 5.1: (a) Signal set. (b) Orthonormal function. (c) Signal space representation.

the two signals can be expressed as

$$s_1(t) = \sqrt{E}\phi_1(t), \tag{5.29}$$

$$s_2(t) = -\sqrt{E}\phi_1(t). \tag{5.30}$$

The geometrical representation of the two signals $s_1(t)$ and $s_2(t)$ is presented in Figure 5.4-(c). Note that the distance between the two signals is $d_{21} = 2\sqrt{E}$.

Example 5.2. The signal set considered in this example is given in Figure 5.5-(a). Again this is a special case because the two signals are *orthogonal*. The energy in each signal is equal to $V^2T_b \equiv E$ (joules). The first orthonormal basis function is

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E}}. \tag{5.31}$$

The correlation coefficient, ρ, is

$$\rho = \int_0^{T_b} \frac{s_1(t)s_2(t)}{E}\mathrm{d}t = 0, \tag{5.32}$$

which shows that the two signals are orthogonal. Therefore $\phi_2^{'}(t) =$

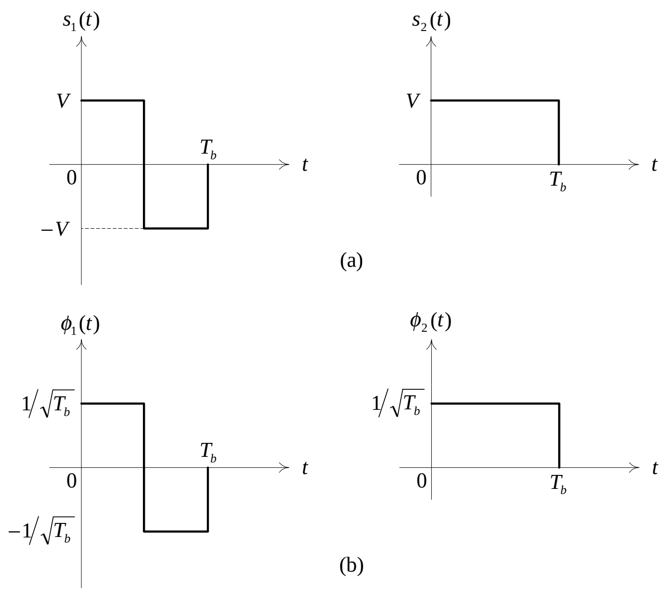


Fig. 5.5. Example 5.2: (a) Signal set. (b) Orthonormal functions.

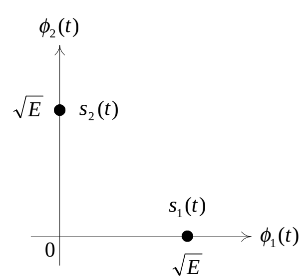


Fig. 5.6. Example 5.2: Signal space representation.

$\frac{s_2(t)}{\sqrt{E}} = \phi_2(t)$. Thus the signals $s_1(t)$ and $s_2(t)$ are expressed as,

$$s_1(t) = \sqrt{E}\phi_1(t), \quad (5.33)$$

$$s_2(t) = \sqrt{E}\phi_2(t). \quad (5.34)$$

Graphically, the orthonormal basis functions $\phi_1(t)$ and $\phi_2(t)$ look as in Figure 5.5-(b) and the signal space is plotted in Figure 5.6. The distance between the two signals can be easily computed as follows:

$$d_{21} = \sqrt{E+E} = \sqrt{2E} = \sqrt{2}\sqrt{E}. \tag{5.35}$$

In comparing Examples 5.1 and 5.2 we observe that the *energy per bit* at the transmitter or sending end is the same in each example. The signals in Example 5.2 however are closer together and therefore at the receiving end, in the presence of noise, we would expect more difficulty in distinguishing which signal was sent. We shall see presently that this is the case and quantitatively express this increased difficulty.

Example 5.3. This is a generalization of Examples 5.1 and 5.2. It is included principally to illustrate the geometrical representation of two signals. The signal set is shown in Figure 5.7, where each signal has energy equal to $E = V^2 T_b$. The first basis function is $\phi_1(t) = \frac{s_1(t)}{\sqrt{E}}$. The correlation coefficient ρ depends on parameter α and it is given by,

$$\rho = \frac{1}{E}\int_0^{T_b} s_2(t)s_1(t)\mathrm{d}t = \frac{1}{V^2T_b}\left[V^2\alpha - V^2(T_b-\alpha)\right] = \frac{2\alpha}{T_b} - 1. \tag{5.36}$$

As a check, for $\alpha = 0$, $\rho = -1$ and for $\alpha = \frac{T_b}{2}$, $\rho = 0$, as expected. The second basis function is

$$\phi_2(t) = \frac{1}{\sqrt{E(1-\rho^2)}}[s_2(t) - \rho s_1(t)]. \tag{5.37}$$

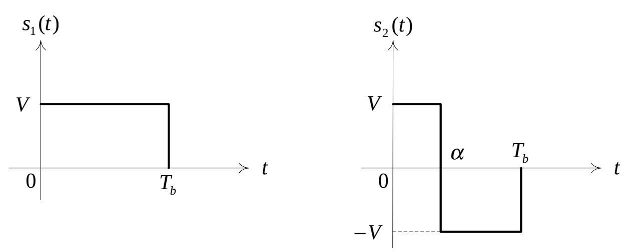


Fig. 5.7. Example 5.3: Signal set.

To obtain the geometrical picture, consider the case when $\alpha = \frac{1}{4}T_b$. For this value of α one has $\rho = -\frac{1}{2}$. As before $\phi_1(t) = \frac{s_1(t)}{\sqrt{E}}$, whereas the second orthonormal function is given by,

$$\phi_2(t) = \frac{2}{\sqrt{3}V\sqrt{T_b}}\left[s_2(t) + \frac{1}{2}s_1(t)\right]. \tag{5.38}$$

The two orthonormal basis functions are plotted in Figure 5.8. The geometrical representation of $s_1(t)$ and $s_2(t)$ is given in Figure 5.9. Note that the coefficients for the representation of $s_2(t)$ are $s_{21} = -\frac{1}{2}\sqrt{E}$ and $s_{22} = \frac{\sqrt{3}}{2}\sqrt{E}$. Since $s_{21}^2 + s_{22}^2 = E$, the signal $s_2(t)$ is $\sqrt{E}$ away from the origin. In general as α varies from 0 to T_b, the function $\phi_2(t)$ changes. However, for each specific $\phi_2(t)$ the signal $s_2(t)$ is at a distance $\sqrt{E}$ from the origin. The locus of $s_2(t)$ is also plotted in Figure 5.9. Note that as α increases, ρ increases and the distance between the two signals decreases.

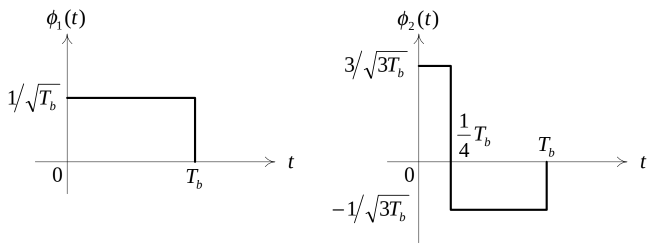


Fig. 5.8. Example 5.3: Orthonormal functions.

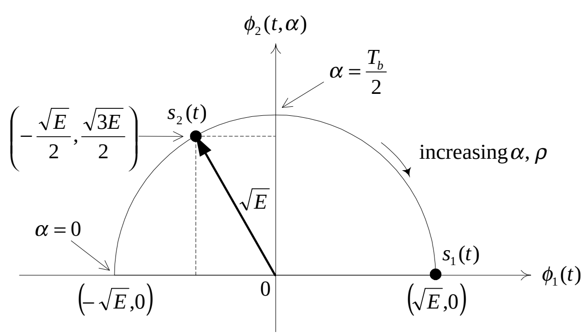


Fig. 5.9. Example 5.3: Signal space representation.

Example 5.4. Consider the signal set shown in Figure 5.10-(a). Again the energy in each signal is $E = V^2 T_b$ joules. We have the following:

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E}}, \tag{5.39}$$

$$\rho = \frac{1}{E}\int_0^{T_b} s_2(t)s_1(t)\mathrm{d}t = \frac{2}{E}\int_0^{T_b/2}\left(\frac{2\sqrt{3}}{T_b}Vt\right)V\mathrm{d}t = \frac{\sqrt{3}}{2}, \quad (5.40)$$

$$\phi_2(t) = \frac{1}{(1-\frac{3}{4})^{\frac{1}{2}}}\left[\frac{s_2(t)}{\sqrt{E}} - \rho\frac{s_1(t)}{\sqrt{E}}\right] = \frac{2}{\sqrt{E}}\left[s_2(t) - \frac{\sqrt{3}}{2}s_1(t)\right], \quad (5.41)$$

$$s_{21} = \frac{\sqrt{3}}{2}\sqrt{E}, \quad s_{22} = \frac{1}{2}\sqrt{E}. \quad (5.42)$$

The distance between the two signals is

$$\begin{aligned} d_{21} &= \left[\int_0^{T_b}[s_2(t)-s_1(t)]^2\mathrm{d}t\right]^{\frac{1}{2}} = \left\{\left[\sqrt{E}\left(1-\frac{\sqrt{3}}{2}\right)\right]^2 + \left(\frac{\sqrt{E}}{2}\right)^2\right\}^{\frac{1}{2}} \\ &= \sqrt{E}\left(2-\sqrt{3}\right)^{\frac{1}{2}} = \sqrt{\left(2-\sqrt{3}\right)E} \end{aligned} \quad (5.43)$$

The orthonormal functions are plotted in Figure 5.10-(b), whereas the signal space representation is illustrated in Figure 5.11.

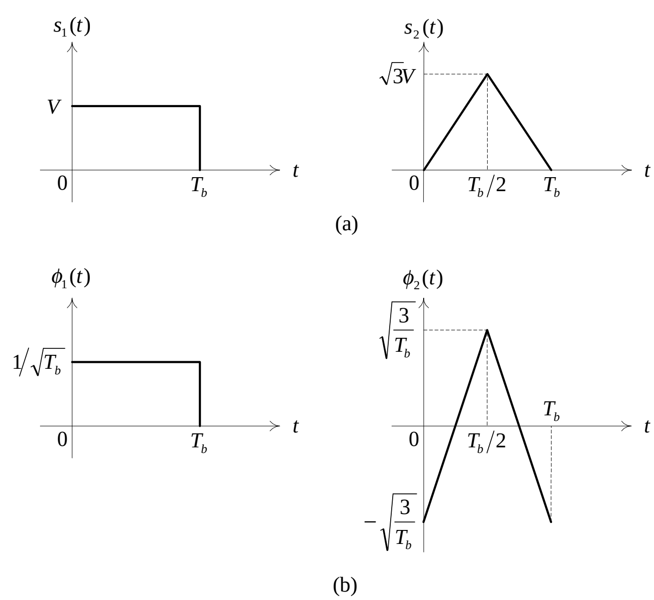


Fig. 5.10. Example 5.4: (a) Signal set. (b) Orthonormal functions.

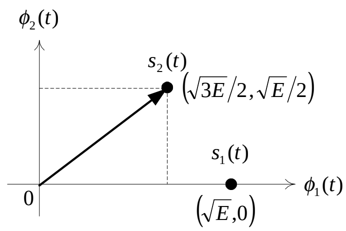


Fig. 5.11. Example 5.4: Signal space representation.

Example 5.5. As a final example to illustrate the Gram-Schmidt procedure, consider two sinusoidal signals of the same frequency but different phase:

$$s_1(t) = \sqrt{E}\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t), \quad (5.44)$$

$$s_2(t) = \sqrt{E}\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t + \theta) \quad (5.45)$$

Choose $f_c = \frac{k}{2T_b}$, k an integer. This choice means that $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t)$ are orthogonal over a duration of T_b seconds. The energy in each signal is

$$E_1 = E\int_0^{T_b} \frac{2}{T_b}\cos^2(2\pi f_c t)\mathrm{d}t = E, \quad (5.46)$$

$$E_2 = E\int_0^{T_b} \frac{2}{T_b}\cos^2(2\pi f_c t + \theta)\mathrm{d}t = E. \quad (5.47)$$

The first orthonormal function is

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E}} = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t). \quad (5.48)$$

Now using the basic trig identity $\cos(x + y) = \cos x \cos y - \sin x \sin y$ write $s_2(t)$ as,

$$s_2(t) = \left(\sqrt{E}\cos\theta\right)\left[\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)\right] + \left(-\sqrt{E}\sin\theta\right)\left[\sqrt{\frac{2}{T_b}}\sin(2\pi f_c t)\right] \quad (5.49)$$

Since we have said that $\sin(2\pi f_c t)$ is orthogonal to $\cos(2\pi f_c t)$ (you might want to show this for yourself) over the interval $(0, T_b)$, our second basis function is chosen to be:

$$\phi_2(t) = \sqrt{\frac{2}{T_b}} \sin(2\pi f_c t). \tag{5.50}$$

Because $s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t)$, by inspection we have

$$s_{21} = \sqrt{E}\cos\theta, \quad s_{22} = -\sqrt{E}\sin\theta. \tag{5.51}$$

Finally, the signal space plot looks as shown in Figure 5.12.

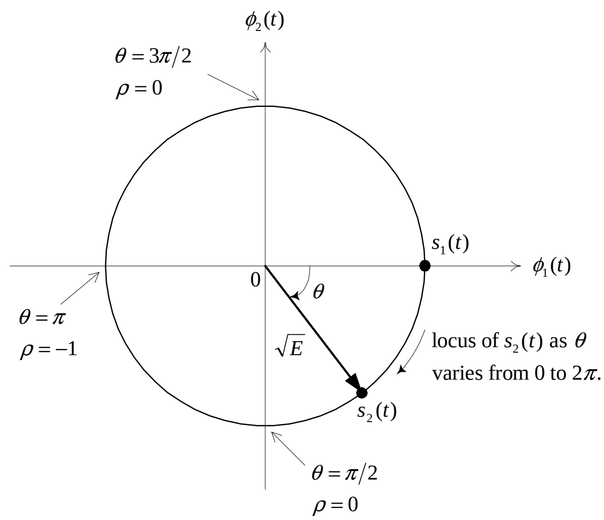


Fig. 5.12. Example 5.5: Signal space representation.

5.2 Representation of the Noise

As seen in the previous section, the signal set $\{s_1(t), s_2(t)\}$ used to transmit the binary data needs at most only two orthonormal functions to be represented exactly. In contrast, to represent the random noise signal, $w(t)$, in the time interval $[(k-1)T_b, kT_b]$ (the interval to receive bit b_k) we need to use a *complete* orthonormal set of known deterministic functions. The series representation of the noise is given by,

$$w(t) = \sum_{i=1}^{\infty} w_i \phi_i(t), \tag{5.52}$$

where the coefficients w_i are given as follows:

$$w_i = \int_0^{T_b} w(t)\phi_i(t)\mathrm{d}t. \tag{5.53}$$

Again the coefficient w_i can be viewed as the projection of the noise process onto the $\phi_i(t)$ axis. Note also that the above representation can be used for any random signal.

The coefficients w_i's are *random variables* and understanding their statistical properties is imperative in developing the optimum receiver. When the noise process $w(t)$ is zero-mean and white, one has the following important properties of these coefficients.†

(i) The average (also known as the mean, or DC, or expected) value of w_i is zero. This is because $w(t)$ is a zero-mean random process. Mathematically,

$$\begin{aligned} \mathcal{E}\{w_i\} &= \mathcal{E}\left\{\int_0^{T_b} w(t)\phi_i(t)\mathrm{d}t\right\} \\ &= \int_0^{T_b} \mathcal{E}\{w(t)\}\phi_i(t)\mathrm{d}t = 0. \end{aligned} \tag{5.54}$$

(ii) The correlation between w_i and w_j is given by,

$$\begin{aligned} \mathcal{E}\{w_i w_j\} &= \mathcal{E}\Big\{\int_0^{T_b} \mathrm{d}\lambda w(\lambda)\phi_i(\lambda)\int_0^{T_b} \mathrm{d}\tau w(\tau)\phi_j(\tau)\Big\} \\ &= \int_0^{T_b} \phi_i(\lambda)\left[\int_0^{T_b} \phi_j(\tau)\mathcal{E}\{w(\lambda)w(\tau)\}\mathrm{d}\tau\right]\mathrm{d}\lambda. \end{aligned} \tag{5.55}$$

But $w(t)$ is *white* noise, therefore $\mathcal{E}\{w(\lambda)w(\tau)\} = \frac{N_0}{2}\delta(\lambda - \tau)$.

† It is important to stress the simple fact that if the noise $w(t)$ is zero-mean and white, it is not necessarily Gaussian.

Equation (5.55) can now be written as,

$$\begin{aligned}\mathcal{E}\{w_i w_j\} &= \int_0^{T_b} \phi_i(\lambda)\left[\int_0^{T_b} \phi_j(\tau)\frac{N_0}{2}\delta(\lambda-\tau)\mathrm{d}\tau\right]\mathrm{d}\lambda \\ &= \frac{N_0}{2}\int_0^{T_b} \phi_i(\lambda)\phi_j(\lambda)\left[\int_0^{T_b} \delta(\lambda-\tau)\mathrm{d}\tau\right]\mathrm{d}\lambda \\ &= \frac{N_0}{2}\int_0^{T_b} \phi_i(\lambda)\phi_j(\lambda)\mathrm{d}\lambda \\ &= \begin{cases} \frac{N_0}{2}, & i=j \\ 0, & i\neq j \end{cases}. \end{aligned} \tag{5.56}$$

The above result means that w_i and w_j are *uncorrelated* if $i \neq j$.

In summary, the coefficients $\{w_1, w_2, \ldots\}$ are zero-mean and uncorrelated random variables.

Now, if the noise process $w(t)$ is not only zero-mean and white, but also Gaussian, then each w_i is a Gaussian random variable. This is because w_i is essentially obtained by passing the Gaussian process $w(t)$ through a linear transformation (system). Because the random variables $\{w_1, w_2, \ldots\}$ are Gaussian and uncorrelated, we deduce the important property that they are *statistically independent*. Furthermore, we note that the above properties do not depend on the set $\{\phi_i(t), i = 1, 2, \ldots\}$ chosen, i.e., on the orthonormal basis functions. The set of orthonormal basis functions that we shall choose will have as the first two functions the functions $\phi_1(t)$ and $\phi_2(t)$ used to represent the two signals $s_1(t)$ and $s_2(t)$ exactly. The remaining functions, i.e., $\phi_3(t)$, $\phi_4(t)$, $\ldots$, are simply chosen to complete the set. However as shall be seen shortly, practically, we do not need to find these functions.

5.3 Optimum Receiver

In any bit interval we receive the noise-corrupted signal. Let us concentrate, without any loss of generality, on the first bit interval, the received signal is

$$\begin{aligned} r(t) &= s_i(t) + w(t), \quad 0 \le t \le T_b \\ &= \begin{cases} s_1(t) + w(t), & \text{if a "0" is transmitted} \\ s_2(t) + w(t), & \text{if a "1" is transmitted} \end{cases}. \end{aligned} \tag{5.57}$$

The following, besides $r(t)$, is known to us. We know exactly when the time interval begins, i.e., the receiver is synchronized with the transmitter. The receiver knows precisely the two signals $s_1(t)$ and $s_2(t)$. Finally the priori probability of a 0 or 1 being transmitted is also known.

To obtain the optimum receiver, we first expand $r(t)$ into a series using the orthonormal functions $\{\phi_1(t), \phi_2(t), \phi_3(t), \ldots\}$. As mentioned before, the first two functions are chosen so that they can represent the signals $s_1(t)$ and $s_2(t)$ exactly (in effect the signals $s_1(t)$ and $s_2(t)$ determine them, for example by applying the Gram-Schmidt procedure). The rest are chosen to complete the orthonormal set. The signal $r(t)$ over the time interval $[0, T_b]$ can therefore be expressed as,

$$\begin{aligned} r(t) &= s_i(t) + w(t), \quad 0 \le t \le T_b \\ &= \underbrace{[s_{i1}\phi_1(t) + s_{i2}\phi_2(t)]}_{s_i(t)} \\ &\quad + \underbrace{[w_1\phi_1(t) + w_2\phi_2(t) + w_3\phi_3(t) + w_4\phi_4(t) + \cdots]}_{w(t)} \\ &= (s_{i1} + w_1)\phi_1(t) + (s_{i2} + w_2)\phi_2(t) + w_3\phi_3(t) + w_4\phi_4(t) + \cdots \\ &= r_1\phi_1(t) + r_2\phi_2(t) + r_3\phi_3(t) + r_4\phi_4(t) + \cdots \end{aligned} \tag{5.58}$$

where $r_j = \int_0^{T_b} r(t)\phi_j(t)\mathrm{d}t$, and

$$\begin{aligned} r_1 &= s_{i1} + w_1 \\ r_2 &= s_{i2} + w_2 \\ r_3 &= w_3 \\ r_4 &= w_4 \\ &\vdots \end{aligned} \tag{5.59}$$

It is important to note that r_j, for $j = 3, 4, 5, \ldots$, does not depend on which signal ($s_1(t)$ or $s_2(t)$) was transmitted.

The decision as to which signal has been transmitted can now be based on the observations $r_1, r_2, r_3, r_4, \ldots$. Before proceeding we need a criterion which the decision should optimize. The criterion that we choose is the perfectly natural one, namely *to minimize the error probability*.

To derive the receiver that minimizes the error probability, consider the set of observations $\mathbf{r} = \{r_1, r_2, r_3, \ldots\}$. If we consider only the first n terms in the set, then they form an n-dimensional observation space which we have to partition into *decision regions* in order to minimize the error probability. This is illustrated in Figure 5.13.

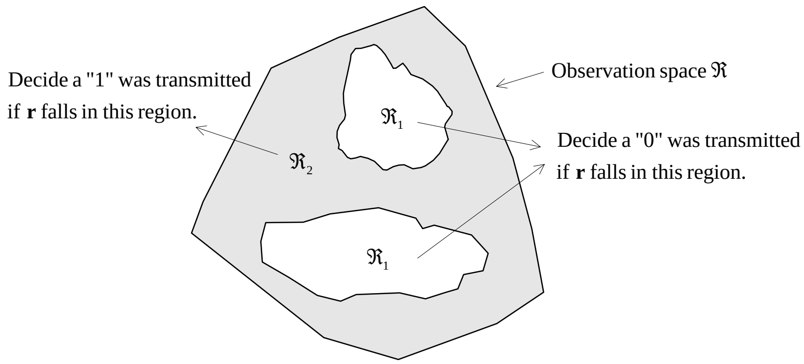


Fig. 5.13. Observation space and decision regions.

The probability of making an error can be expressed as,

$$\begin{aligned}\Pr[\text{error}] = \Pr[(&\text{“0” decided and “1” transmitted}) \text{ or} \\ &(\text{“1” decided and “0” transmitted})].\end{aligned} \tag{5.60}$$

Using obvious notations and the fact that the two events that constitute the error event are *mutually exclusive*, the above can be written as,

$$\begin{aligned}\Pr[\text{error}] &= \Pr[0_D, 1_T] + \Pr[1_D, 0_T] \\ &= \Pr[0_D|1_T]\Pr[1_T] + \Pr[1_D|0_T]\Pr[0_T],\end{aligned} \tag{5.61}$$

where the second equality follows from Bayes' rule. Consider now the quantity $\Pr[0_D|1_T]$. A zero is decided when the observation **r** falls into region $\mathcal{R}_1$. Therefore the probability that a "0" is decided given that a "1" was transmitted is the same as the probability that **r** falls in region $\mathcal{R}_1$ when a "1" is transmitted. But this latter probability is given by the volume under the conditional density $f(\mathbf{r}|1_T)$ in region $\mathcal{R}_1$, i.e.,

$$\Pr[0_D|1_T] = \int_{\mathcal{R}_1} f(\mathbf{r}|1_T)\mathrm{d}\mathbf{r} \quad \text{(note that this is a \textit{multiple} integral).} \tag{5.62}$$

Since $\Pr[0_T] = P_1$ and $\Pr[1_T] = P_2$, one has

$$\begin{aligned} \Pr[\text{error}] &= P_2 \int_{\mathcal{R}_1} f(\mathbf{r}|1_T)\mathrm{d}\mathbf{r} + P_1 \int_{\mathcal{R}_2} f(\mathbf{r}|0_T)\mathrm{d}\mathbf{r} \\ &= P_2 \int_{\mathcal{R}-\mathcal{R}_2} f(\mathbf{r}|1_T)\mathrm{d}\mathbf{r} + P_1 \int_{\mathcal{R}_2} f(\mathbf{r}|0_T)\mathrm{d}\mathbf{r} \\ &= P_2 \int_{\mathcal{R}} f(\mathbf{r}|1_T)\mathrm{d}\mathbf{r} + \int_{\mathcal{R}_2} [P_1 f(\mathbf{r}|0_T) - P_2 f(\mathbf{r}|1_T)]\mathrm{d}\mathbf{r} \\ &= P_2 + \int_{\mathcal{R}_2} [P_1 f(\mathbf{r}|0_T) - P_2 f(\mathbf{r}|1_T)]\mathrm{d}\mathbf{r} \end{aligned} \quad (5.63)$$

The error of course depends on how the observation space is partitioned. Looking at the above expression for the error probability we observe that if each $\mathbf{r}$ that makes the integrand $[P_1 f(\mathbf{r}|0_T) - P_2 f(\mathbf{r}|1_T)]$ negative is assigned to $\mathcal{R}_2$, then the error probability is minimized. Note also that, for the observation $\mathbf{r}$ that gives $[P_1 f(\mathbf{r}|0_T) - P_2 f(\mathbf{r}|1_T)] = 0$, it does not matter to put $\mathbf{r}$ into $\mathcal{R}_1$ or $\mathcal{R}_2$, i.e., the receiver can arbitrarily decide "1" or "0". Thus the minimum error probability decision rule can be expressed as,

$$\begin{cases} P_1 f(\mathbf{r}|0_T) - P_2 f(\mathbf{r}|1_T) \geq 0 & \Rightarrow \text{ decide "0" } (0_D) \\ P_1 f(\mathbf{r}|0_T) - P_2 f(\mathbf{r}|1_T) < 0 & \Rightarrow \text{ decide "1" } (1_D) \end{cases}. \quad (5.64)$$

Equivalently,

$$\frac{f(\mathbf{r}|1_T)}{f(\mathbf{r}|0_T)} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{P_1}{P_2}. \quad (5.65)$$

The expression $\frac{f(\mathbf{r}|1_T)}{f(\mathbf{r}|0_T)}$ is commonly called the *likelihood ratio*. The receiver consists of computing this ratio and comparing the result to a threshold determined by the a prior probabilities. Furthermore, observe that the decision rule in (5.65) was derived without specifying any statistical properties of the noise process $w(t)$. In other words, it is true for any set of observations $\mathbf{r}$.

The decision rule in (5.65) can, however, be greatly simplified when the noise $w(t)$ is zero-mean, white and Gaussian. In this case, the observations $r_1, r_2, r_3, r_4, \ldots$, are statistically independent Gaussian random variables. The variance of each coefficient is equal to the variance of the noise coefficient, w_j, which is $N_0/2$ (watts). The average or expected values of the first two coefficients r_1 and r_2 are s_{i1} and s_{i2}, respectively. The other coefficients, $r_j, j \geq 3$, have zero mean. The *conditional* *densities* $f(\mathbf{r}|1_T)$ and $f(\mathbf{r}|0_T)$ can simply be written as products of the individual probability density functions as follows:

$$f(\mathbf{r}|1_T) = f(r_1|1_T)f(r_2|1_T)f(r_3|1_T)\cdots f(r_j|1_T),\cdots \quad (5.66)$$

$$f(\mathbf{r}|0_T) = f(r_1|0_T)f(r_2|0_T)f(r_3|0_T)\cdots f(r_j|0_T)\cdots \quad (5.67)$$

or

$$f(\mathbf{r}|1_T) = \frac{1}{\sqrt{\pi N_0}}\exp\left[-\frac{(r_1-s_{21})^2}{N_0}\right]\frac{1}{\sqrt{\pi N_0}}\exp\left[-\frac{(r_2-s_{22})^2}{N_0}\right] \times f(r_3|1_T)\cdots \quad (5.68)$$

$$f(\mathbf{r}|0_T) = \frac{1}{\sqrt{\pi N_0}}\exp\left[-\frac{(r_1-s_{11})^2}{N_0}\right]\frac{1}{\sqrt{\pi N_0}}\exp\left[-\frac{(r_2-s_{12})^2}{N_0}\right] \times f(r_3|0_T)\cdots \quad (5.69)$$

Note that the terms $f(r_j|1_T) = f(r_j|0_T) = \frac{1}{\sqrt{\pi N_0}}\exp\left(-r_j^2/N_0\right)$ for $j \geq 3$. Therefore they will cancel in the likelihood ratio in (5.65). Thus the decision rule becomes

$$\frac{\exp\left[-(r_1-s_{21})^2/N_0\right]\exp\left[-(r_2-s_{22})^2/N_0\right]}{\exp\left[-(r_1-s_{11})^2/N_0\right]\exp\left[-(r_2-s_{12})^2/N_0\right]} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{P_1}{P_2} \quad (5.70)$$

To further simplify the expression, take the natural logarithm of both sides of (5.70). Since the natural logarithm is a *monotonic* function, the inequality is preserved for each (r_1, r_2) pair. The resulting decision rule becomes

$$(r_1-s_{11})^2 + (r_2-s_{12})^2 \underset{0_D}{\overset{1_D}{\gtrless}} (r_1-s_{21})^2 + (r_2-s_{22})^2 + N_0 \ln\left(\frac{P_1}{P_2}\right) \quad (5.71)$$

The above rule has an interesting geometrical interpretation. The quantity $(r_1-s_{11})^2 + (r_2-s_{12})^2$ is the *distance squared* from the projection (r_1, r_2) of the received signal $r(t)$ to the transmitted signal $s_1(t)$. Similarly, $(r_1-s_{21})^2 + (r_2-s_{22})^2$ is the distance squared from (r_1, r_2) to $s_2(t)$. For the special case of $P_1 = P_2$, i.e., each signal is equally likely, the decision rule becomes

$$(r_1-s_{11})^2 + (r_2-s_{12})^2 \underset{0_D}{\overset{1_D}{\gtrless}} (r_1-s_{21})^2 + (r_2-s_{22})^2 \quad (5.72)$$

In essence, the above implies that the optimum receiver needs to determine the distance from $r(t)$ to both $s_1(t)$ and $s_2(t)$ and then choose the signal that $r(t)$ is closest to.

For the more general case of $P_1 \neq P_2$ the decision regions are determined by a line *perpendicular* to the line joining the two signals $\{s_1(t), s_2(t)\}$, except that it now shifts toward $s_2(t)$ (if $P_1 > P_2$) or toward $s_1(t)$ (if $P_1 < P_2$). This is illustrated in Figure 5.14.

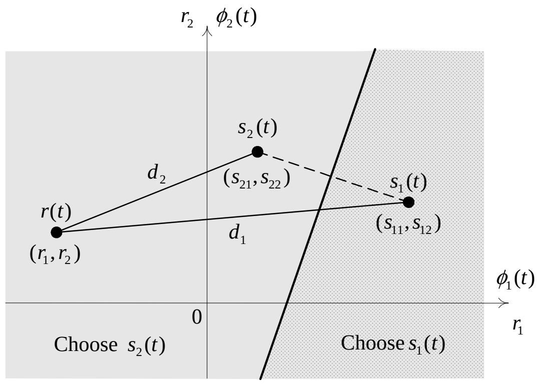


Fig. 5.14. Decision regions are determined by a line perpendicular to the line joining $s_1(t)$ to $s_2(t)$.

5.4 Receiver Implementation

To determine, in a minimum error probability sense, whether a "0" or "1" was transmitted we need to determine (r_1, r_2) and then use (5.71) to make a decision. The receiver would therefore look as shown in Figure 5.15.

The process of multiplying $r(t)$ by $\phi_1(t)$ and integrating over the bit duration is that of *correlation* and therefore the above is called a correlation receiver configuration. Since at the end of each bit duration the integrator must be reset to zero initial condition, the above is also commonly called an integrate-and-dump receiver. Moreover, the decision block can be simplified somewhat by expanding the terms in (5.71) to arrive at

$$\begin{array}{ccc} (r_1 - s_{11})^2 + (r_2 - s_{12})^2 & \overset{1_D}{\underset{0_D}{\gtrless}} & (r_1 - s_{21})^2 + (r_2 - s_{22})^2 \\ -N_0 \ln P_1 & & -N_0 \ln P_2 \end{array} \tag{5.73}$$

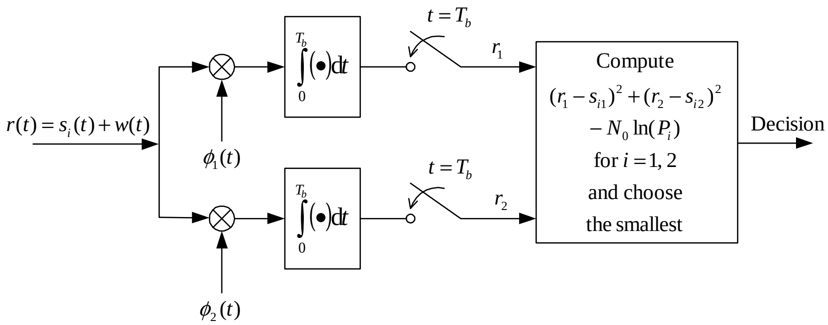


Fig. 5.15. Correlation receiver implementation.

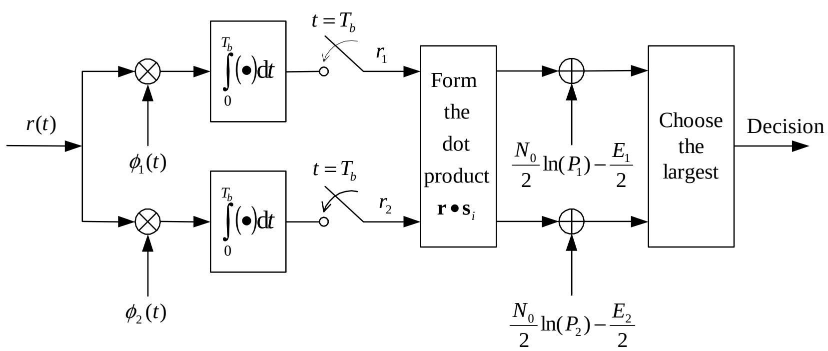


Fig. 5.16. A different implementation of a correlation receiver.

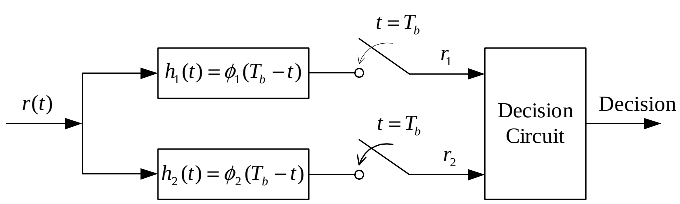


Fig. 5.17. Receiver implementation using matched filters.

$$\begin{array}{ccc} r_1^2 - 2r_1 s_{11} + s_{11}^2 + r_2^2 & \underset{0_D}{\overset{1_D}{\gtrless}} & r_1^2 - 2r_1 s_{21} + s_{21}^2 + r_2^2 \\ -2r_2 s_{12} + s_{12}^2 - N_0 \ln P_1 & & -2r_2 s_{22} + s_{22}^2 - N_0 \ln P_2 \end{array} \quad (5.74)$$

Cancelling the common terms, recognizing that $E_1 = s_{11}^2 + s_{12}^2$, $E_2 =$

$s_{21}^2 + s_{22}^2$ and rearranging yields

$$r_1 s_{21} + r_2 s_{22} - \frac{E_2}{2} + \frac{N_0}{2} \ln P_2 \underset{0_D}{\overset{1_D}{\gtrless}} r_1 s_{11} + r_2 s_{12} - \frac{E_1}{2} + \frac{N_0}{2} \ln P_1 \tag{5.75}$$

The term $r_1 s_{i1} + r_2 s_{i2} = (r_1, r_2)\begin{pmatrix} s_{i2} \\ s_{i1} \end{pmatrix}$ can be interpreted as the dot product between the vector $\mathbf{r} = (r_1, r_2)$ representing the received signal $r(t)$ and the vector $\mathbf{s}_i = (s_{i1}, s_{i2})$ representing the signal $s_i(t)$. Therefore the receiver implementation can be redrawn as in Figure 5.16.

The correlation part of the optimum receiver involves a multiplier and integrator. Multipliers are difficult to realize physically. As an alternative, the correlator section can be implemented by a finite impulse response filter as shown in Figure 5.17, where

$$h_1(t) = \phi_1(T_b - t), \tag{5.76}$$

$$h_2(t) = \phi_2(T_b - t). \tag{5.77}$$

The above filters are generally referred to as *matched filters*, in this case they are "matched" to the basis functions.

Example 5.6. Consider the signal set $\{s_1(t), s_2(t)\}$ shown in Figure 5.18-(a). Simple inspection shows that this signal set can be represented by the orthonormal functions $\{\phi_1(t), \phi_2(t)\}$ in Figure 5.18-(b) as follows:

$$\begin{aligned} s_1(t) &= \phi_1(t) + \frac{1}{2}\phi_2(t) \\ s_2(t) &= -\phi_1(t) + \phi_2(t) \end{aligned}$$

The energy in each of the signals is given as:

$$E_1 = \int_0^{T_b} s_1^2(t)\mathrm{d}t = 1.25 \text{ (joules)}, \quad E_2 = \int_0^{T_b} s_2^2(t)\mathrm{d}t = 2 \text{ (joules)}. \tag{5.78}$$

Unlike the examples considered up to now, here the two signals have unequal energy. The signal space plot looks as shown in Figure 5.19. Consider now the detection of the transmitted signal over a bit interval where the power spectral density of the white Gaussian noise is $\frac{N_0}{2} = 0.5$ (watts/Hz). The optimum receiver to minimize error consists of projecting the received signal $r(t) = s_i(t) + w(t)$ onto $\phi_1(t), \phi_2(t)$ and

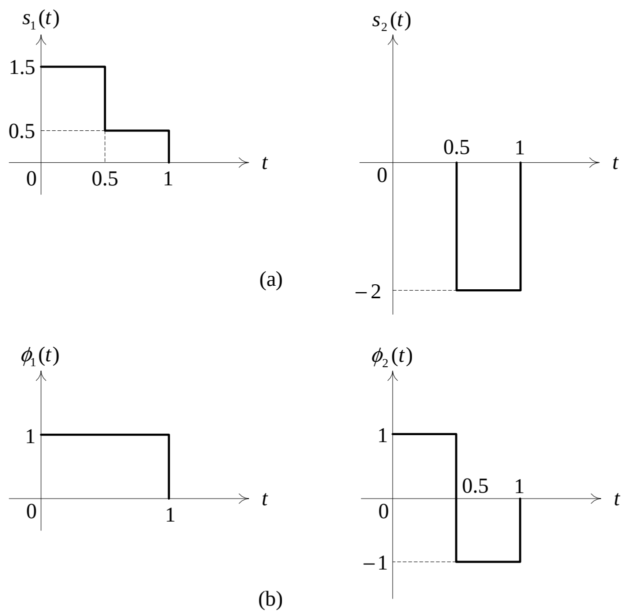


Fig. 5.18. Example 5.6: (a) Signal set. (b) Orthonormal functions.

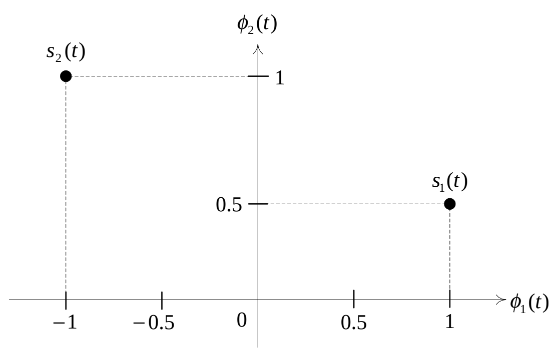


Fig. 5.19. Example 5.6: Signal space representation.

then applying the decision rule given by (5.71), that is

$$(r_1 - 1)^2 + (r_2 - \tfrac{1}{2})^2 \underset{0_D}{\overset{1_D}{\gtrless}} (r_1 + 1)^2 + (r_2 - 1)^2 + \ln\left(\frac{P_1}{P_2}\right) \quad (5.79)$$

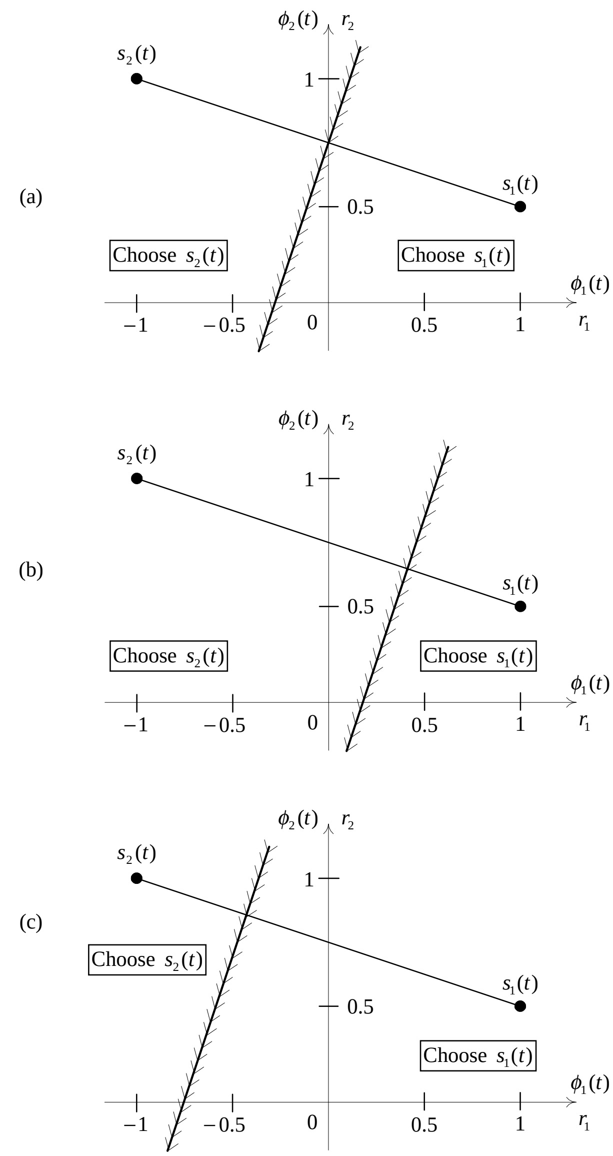


Fig. 5.20. Example 5.6: Decision regions: (a) $P_1 = P_2 = 0.5$. (b) $P_1 = 0.25,\ P_2 = 0.75$. (c) $P_1 = 0.75,\ P_2 = 0.25$.

Upon expanding the decision rule becomes that of (5.75) which can be written as

$$-4r_1 + r_2 - \left(\frac{3}{4} + \ln\frac{P_1}{P_2}\right) \underset{0_D}{\overset{1_D}{\gtrless}} 0 \tag{5.80}$$

The boundary between the two decision regions is given by

$$4r_1 - r_2 + \left(\frac{3}{4} - \ln\frac{P_1}{P_2}\right) = 0 \tag{5.81}$$

which is an equation of a straight line of slope 4 and intercept $\left(\frac{3}{4} - \ln\frac{P_1}{P_2}\right)$. In terms of (r_2, r_1) the equation of the straight line joining the $s_2(t)$ and $s_1(t)$ points is given by

$$\frac{r_2 - s_{12}}{s_{22} - s_{12}} = \frac{r_1 - s_{11}}{s_{21} - s_{11}}, \quad \text{or} \quad r_2 = -\frac{1}{4}r_1 + \frac{3}{4} \tag{5.82}$$

It can be seen that the straight lines defined by (5.81) and (5.82) are perpendicular to each other. The decision regions therefore look as in Figure 5.20 for 3 different sets of a priori probabilities.

Example 5.7. For the second example, consider the two orthonormal functions shown in Figure 5.21. Let the signal set be as follows:

$$s_2(t) = \phi_1(t) + \phi_2(t), \tag{5.83}$$

$$s_1(t) = \phi_1(t) - \phi_2(t). \tag{5.84}$$

The two signals have equal energy of $E_1 = E_2 = 2$ (joules). Note that $s_1(t)$ and $s_2(t)$ have a common component $\phi_1(t)$. Thus, intuitively we would expect that only the component along $\phi_2(t)$ will help us to distinguish between the possible transmitted signals in the presence of noise. This is verified formally by applying (5.71) as follows:

$$(r_1 - 1)^2 + (r_2 + 1)^2 - N_0 \ln P_1 \underset{0_D}{\overset{1_D}{\gtrless}} (r_1 - 1)^2 + (r_2 - 1)^2 - N_0 \ln P_2, \tag{5.85}$$

which simplifies to

$$r_2 \underset{0_D}{\overset{1_D}{\gtrless}} \frac{N_0}{4} \ln\left(\frac{P_1}{P_2}\right) \tag{5.86}$$

The signal set and the decision region are shown in Figure 5.22. The

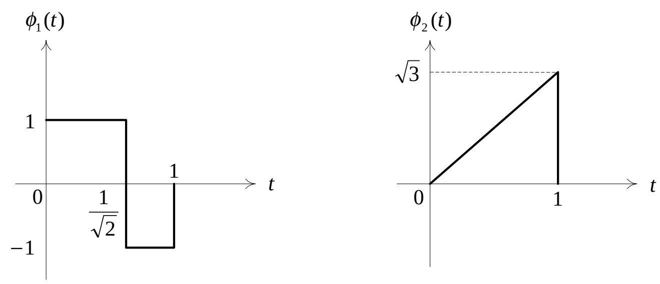


Fig. 5.21. Example 5.7: Orthonormal functions.

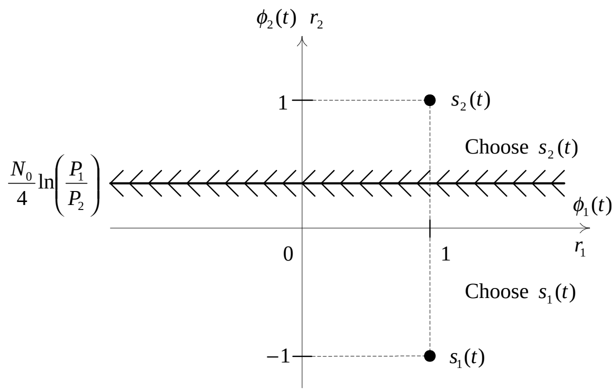


Fig. 5.22. Example 5.7: Signal space representation and the decision regions.

receiver can be implemented using a correlator or a matched filter, as shown in Figures 5.23-(a) and 5.23-(b), respectively.

5.5 Receiver Implementation with only One Correlator or One Matched Filter

In general, for two arbitrary signals $s_1(t)$ and $s_2(t)$ we need two orthonormal basis functions to represent them exactly. Furthermore, the optimum receiver requires the projection of $r(t)$ onto the two basis functions. This in turn means that in the receiver implementation either two correlators or two matched filters are required. In Example 5.7, however, only one correlator or matched filter was required. For binary data transmission, where the transmitted symbol is represented by one

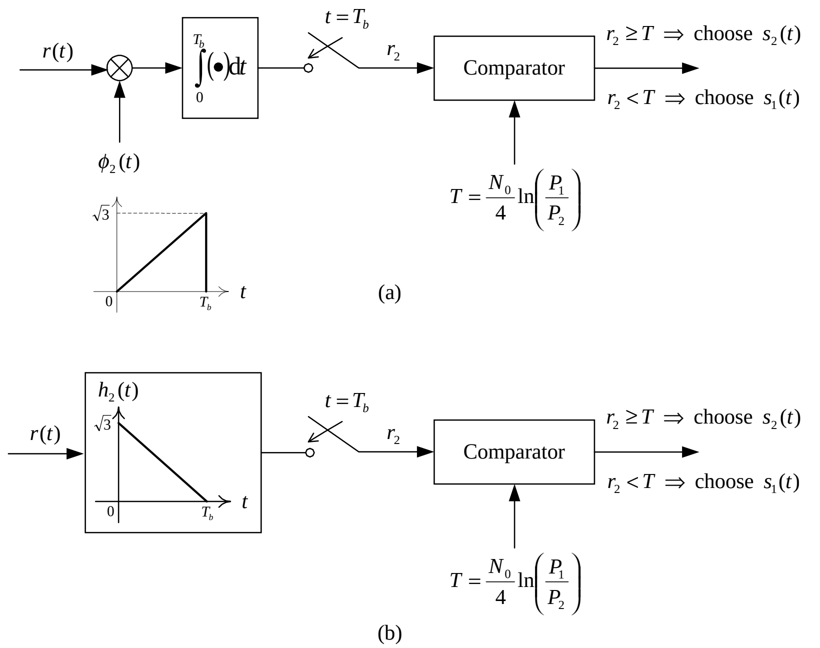


Fig. 5.23. Example 5.7: Receiver implementation. (a) Correlation receiver, and (b) Matched filter receiver.

of two signals, this simplification is *always possible* by a judicious choice of the orthonormal basis.

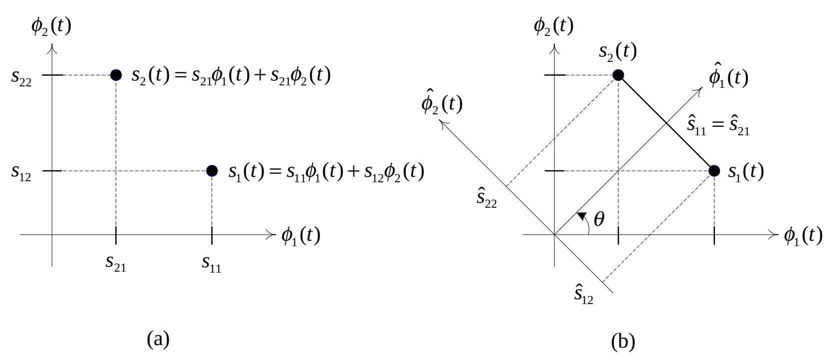


Fig. 5.24. Signal space representation: (a) by $\phi_1(t)$ and $\phi_2(t)$ and (b) by $\hat{\phi}_1(t)$ and $\hat{\phi}_2(t)$.

Consider that $s_1(t)$ and $s_1(t)$ are represented by the orthonormal basis $\phi_1(t)$ and $\phi_2(t)$ as shown in Figure 5.24-(a). To simplify the receiver so that only one correlator or matched filter is needed what we need is to find orthonormal basis functions $\hat{\phi}_1(t)$ and $\hat{\phi}_2(t)$ such that along one axis, say $\hat{\phi}_1(t)$, the signals $s_1(t)$ and $s_2(t)$ have an identical component. To determine this basis set we rotate $\phi_1(t)$ and $\phi_2(t)$ through an angle θ until one of the axis is perpendicular to the line joining $s_1(t)$ to $s_2(t)$ (see Problem 5.2). This rotation can be expressed as,

$$\begin{pmatrix} \hat{\phi}_1(t) \\ \hat{\phi}_2(t) \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix}. \tag{5.87}$$

Now we project the received signal $r(t) = s_i(t) + w(t)$ onto the orthonormal basis functions $\hat{\phi}_1(t)$ and $\hat{\phi}_2(t)$. The components of $s_1(t)$ and $s_2(t)$ along $\hat{\phi}_1(t)$, namely $\hat{s}_{11}$ and $\hat{s}_{21}$, are identical. The noise projections $\hat{w}_1$ and $\hat{w}_2$ are again statistically independent zero-mean Gaussian random variables with variance $N_0/2$.

As before, the other orthonormal basis functions $\hat{\phi}_3(t), \hat{\phi}_4(t), \ldots$, are simply chosen to complete the set. Again, given the fact that $w(t)$ is white and Gaussian, the likelihood ratio of the projections of $r(t)$ onto $\hat{\phi}_1(t), \hat{\phi}_2(t), \hat{\phi}_3(t), \ldots$ is

$$\frac{f(\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots, |1_T)}{f(\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots, |0_T)} = \frac{f(\hat{s}_{21} + \hat{w}_1) f(\hat{s}_{22} + \hat{w}_2) f(\hat{w}_3) \ldots}{f(\hat{s}_{11} + \hat{w}_1) f(\hat{s}_{12} + \hat{w}_2) f(\hat{w}_3) \ldots} \quad \overset{1_D}{\underset{0_D}{\gtrless}} \quad \frac{P_1}{P_2}. \tag{5.88}$$

Which upon cancelling the common terms reduces to

$$\frac{f(\hat{r}_2|1_T)}{f(\hat{r}_2|0_T)} = \frac{f(\hat{s}_{22} + \hat{w}_2)}{f(\hat{s}_{12} + \hat{w}_2)} \quad \overset{1_D}{\underset{0_D}{\gtrless}} \quad \frac{P_1}{P_2}. \tag{5.89}$$

Substituting in the exact expressions of the density functions, the decision rule becomes

$$\frac{(\pi N_0)^{-\frac{1}{2}} \exp[-(\hat{r}_2 - \hat{s}_{22})^2/N_0]}{(\pi N_0)^{-\frac{1}{2}} \exp[-(\hat{r}_2 - \hat{s}_{12})^2/N_0]} \quad \overset{1_D}{\underset{0_D}{\gtrless}} \quad \frac{P_1}{P_2}. \tag{5.90}$$

Taking the natural logarithm and simplifying, the final decision rule is

$$\hat{r}_2 \underset{0_D}{\overset{1_D}{\gtrless}} \frac{\hat{s}_{22}+\hat{s}_{12}}{2} + \left(\frac{N_0/2}{\hat{s}_{22}-\hat{s}_{12}}\right)\ln\left(\frac{P_1}{P_2}\right). \qquad (5.91)$$

Note that, to arrive at the above decision rule, it was assumed that $(\hat{s}_{22}-\hat{s}_{12}) > 0$. This is indeed the case since, as will be shown later on page 110, $(\hat{s}_{22}-\hat{s}_{12})$ measures the distance between the two signals.

Thus the optimum receiver consists of finding $\hat{r}_2$ by projecting $r(t)$ onto $\hat{\phi}_2(t)$, i.e., $\hat{r}_2 = \int_0^{T_b} r(t)\hat{\phi}_2(t)\mathrm{d}t$, and comparing $\hat{r}_2$ to a threshold,

$$T \equiv \frac{\hat{s}_{22}+\hat{s}_{12}}{2} + \left(\frac{N_0/2}{\hat{s}_{22}-\hat{s}_{12}}\right)\ln\left(\frac{P_1}{P_2}\right) \qquad (5.92)$$

Two receiver implementations are shown in Figure 5.25-(a) and 5.25-(b), corresponding to the correlation receiver and the matched filter, respectively.

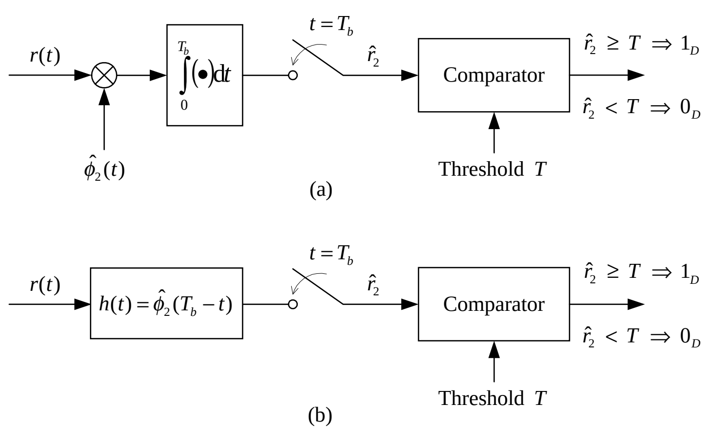


Fig. 5.25. Simplified optimum receiver: (a) Using one correlator. (b) Using one matched filter.

As evidenced from Figure 5.25, one needs to know $\hat{\phi}_2(t)$ in order to implement the optimum receiver. This basis function can be determined as follows. Observe first that $\hat{\phi}_2(t)$ points along the line joining $s_1(t)$ to $s_2(t)$ which, as a vector, is $s_2(t) - s_1(t)$. Thus normalizing this vector so that the resultant time function has unit energy gives $\hat{\phi}_2(t)$. That is,

$$\hat{\phi}_2(t) = \frac{s_2(t) - s_1(t)}{\left\{\int_0^{T_b} [s_2(t) - s_1(t)]^2 \mathrm{d}t\right\}^{\frac{1}{2}}} = \frac{s_2(t) - s_1(t)}{(E_2 - 2\rho\sqrt{E_1 E_2} + E_1)^{\frac{1}{2}}} \quad (5.93)$$

Observe that the above expression of $\hat{\phi}_2(t)$ illustrates that only the difference between the two signals is important in making the optimum decision at the receiver.

Example 5.8. Consider the signal set of Example 5.2. The signal space diagram is shown in Figure 5.26. By inspection, a rotation of 45^0 results in the desired orthonormal basis functions, where $\hat{\phi}_1(t)$ and $\hat{\phi}_2(t)$ can be written as

$$\begin{aligned} \hat{\phi}_1(t) &= \frac{1}{\sqrt{2}}[\phi_1(t) + \phi_2(t)] \\ \hat{\phi}_2(t) &= \frac{1}{\sqrt{2}}[-\phi_1(t) + \phi_2(t)] \end{aligned}$$

The receiver implementations are illustrated in Figures 5.27-(a) and 5.27-(b).

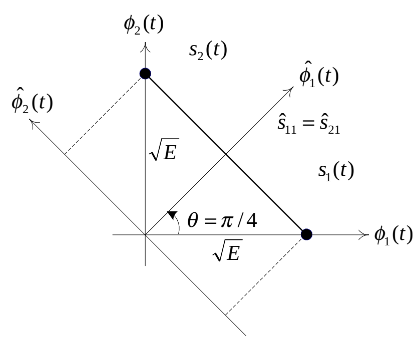


Fig. 5.26. Example 5.8: Signal space representation.

5.6 Receiver Performance

Consider now the determination of the performance of the optimum receiver, which is measured in terms of its error probability. Recall from the preceding section that the detection of the information bit b_k

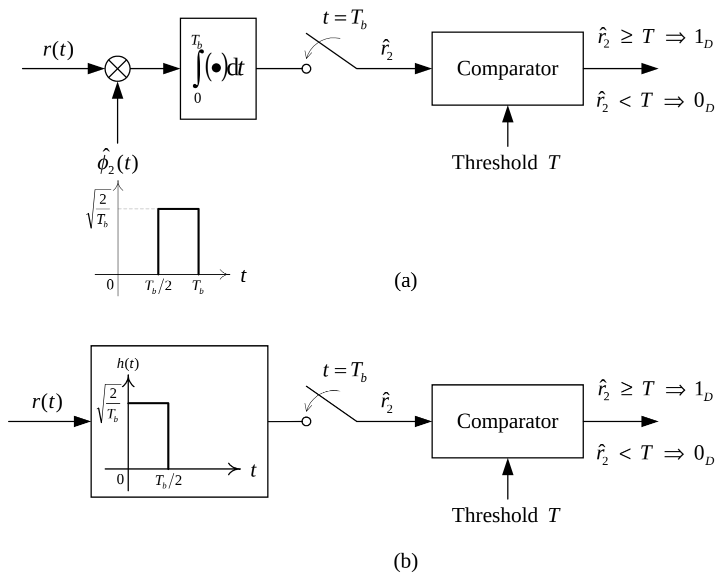


Fig. 5.27. Receiver implementations of Example 5.8: (a) Correlation receiver. (b) Matched filter receiver.

transmitted over the kth interval $[(k-1)T_b, kT_b]$ consists of computing $\hat{r}_2 = \int_{(k-1)T_b}^{kT_b} r(t)\hat{\phi}_2(t)\mathrm{d}t$ and comparing $\hat{r}_2$ to the threshold

$$T = \frac{\hat{s}_{12} + \hat{s}_{22}}{2} + \frac{N_0}{2(\hat{s}_{22} - \hat{s}_{12})} \ln\left(\frac{P_1}{P_2}\right). \tag{5.94}$$

When a "0" is transmitted, $\hat{r}_2$ is a Gaussian random variable of mean $\hat{s}_{12}$ and variance $N_0/2$ (watts). When a "1" is transmitted $\hat{r}_2$ is a Gaussian random variable of mean $\hat{s}_{22}$ and variance $N_0/2$ (watts). Graphically the conditional density functions $f(\hat{r}_2|0_T)$ and $f(\hat{r}_2|1_T)$ are illustrated in Figure 5.28.

The probability of making an error is given by,

$$\begin{aligned} \Pr[\text{error}] &= \Pr[(0 \text{ transmitted and } 1 \text{ decided}) \text{ or} \\ &\quad (1 \text{ transmitted and } 0 \text{ decided})] \\ &= \Pr[(0_T, 1_D) \text{ or } (1_T, 0_D)]. \end{aligned} \tag{5.95}$$

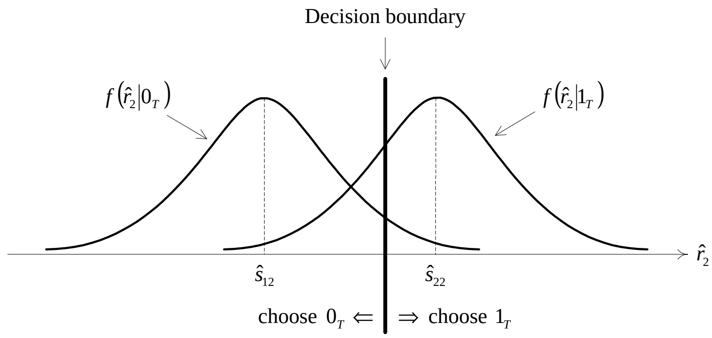


Fig. 5.28. Conditional density functions.

Since the two events are mutually exclusive, one has

$$\begin{aligned}\Pr[\text{error}] &= \Pr[0_T, 1_D] + \Pr[1_T, 0_D] \\ &= \Pr[1_D|0_T]\Pr[0_T] + \Pr[0_D|1_T]\Pr[1_T] \\ &= P_1 \underbrace{\int_T^{\infty} f(\hat{r}_2|0_T)\mathrm{d}\hat{r}_2}_{\text{Area } \mathbf{B}} + P_2 \underbrace{\int_{-\infty}^{T} f(\hat{r}_2|1_T)\mathrm{d}\hat{r}_2}_{\text{Area } \mathbf{A}} \qquad (5.96)\end{aligned}$$

Note that the two integrals in (5.96) are equal to the area **B** and area **A** in Figure 5.29, respectively. Thus the error probability can be calculated as follows:

$$\begin{aligned}\Pr[\text{error}] &= P_1 \int_T^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{(\hat{r}_2 - \hat{s}_{12})^2}{N_0}\right\} \mathrm{d}\hat{r}_2 \\ &+ P_2 \int_{-\infty}^{T} \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{(\hat{r}_2 - \hat{s}_{22})^2}{N_0}\right\} \mathrm{d}\hat{r}_2 \qquad (5.97)\end{aligned}$$

Consider the first integral in (5.97). Change the variables $\lambda \equiv (\hat{r}_2 - \hat{s}_{12})/\sqrt{N_0/2}$, then $\mathrm{d}\lambda = \dfrac{\mathrm{d}\hat{r}_2}{\sqrt{N_0/2}}$ and the lower limit becomes $(T - \hat{s}_{12})/\sqrt{N_0/2}$. The integral therefore can be rewritten as

$$\begin{aligned}\int_T^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{(\hat{r}_2 - \hat{s}_{12})^2}{N_0}\right\} \mathrm{d}\hat{r}_2 &= \frac{1}{\sqrt{2\pi}} \int_{\frac{T-\hat{s}_{12}}{\sqrt{N_0/2}}}^{\infty} \exp\left(-\frac{\lambda^2}{2}\right) \mathrm{d}\lambda \\ &= Q\left(\frac{T - \hat{s}_{12}}{\sqrt{N_0/2}}\right) \qquad (5.98)\end{aligned}$$

where $Q(x)$ is called the *Q-function.* This function is defined as the

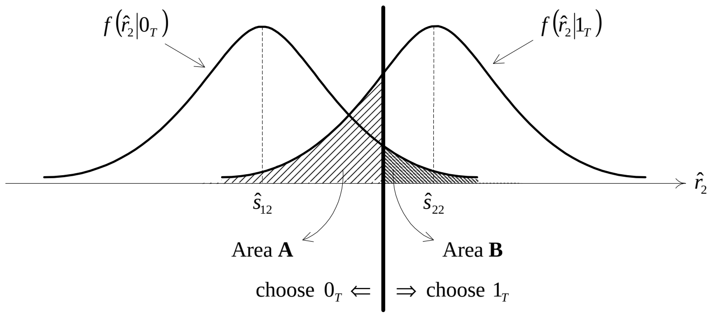


Fig. 5.29. Evaluation of the integrals by areas A and B.

area under a zero-mean, unit-variance Gaussian curve from x to ∞ (see Figure 5.30 for a graphical interpretation). Mathematically,

$$Q(x) \equiv \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left(-\frac{\lambda^2}{2}\right) \mathrm{d}\lambda, \quad -\infty \leq x \leq \infty. \tag{5.99}$$

A plot of $Q(x)$ is shown in Figure 5.31. It is also simple to verify the following useful property of the Q-function:

$$Q(x) = 1 - Q(-x), \quad -\infty \leq x \leq \infty. \tag{5.100}$$

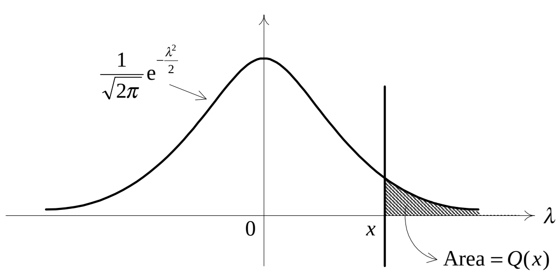


Fig. 5.30. Illustration of the Q-function.

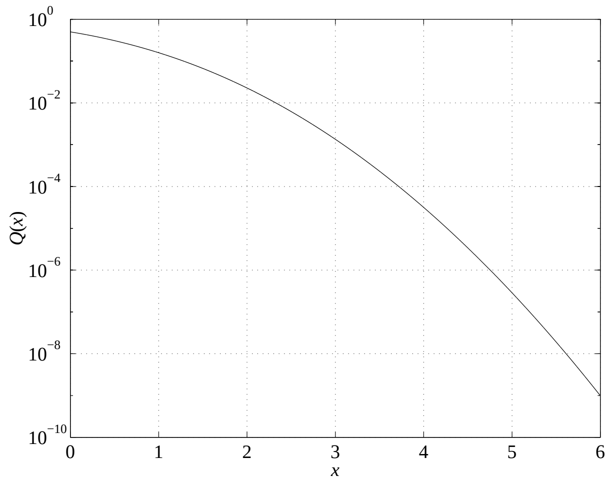


Fig. 5.31. A plot of $Q(x)$.

Similarly, the second integral in (5.97) can be evaluated as follows:

$$\begin{aligned}
&\int_{-\infty}^{T} \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{(\hat{r}_2-\hat{s}_{22})^2}{N_0}\right\} \mathrm{d}\hat{r}_2 \\
&\quad = 1 - \int_{T}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\left\{-\frac{(\hat{r}_2-\hat{s}_{22})^2}{N_0}\right\} \mathrm{d}\hat{r}_2 \\
&\quad = 1 - \frac{1}{\sqrt{2\pi}} \int_{\frac{T-\hat{s}_{22}}{\sqrt{N_0/2}}}^{\infty} \exp\left(-\frac{\lambda^2}{2}\right) \mathrm{d}\lambda \\
&\quad = 1 - Q\left(\frac{T-\hat{s}_{22}}{\sqrt{N_0/2}}\right), \qquad (5.101)
\end{aligned}$$

where the second equality in the above evaluation follows by changing $\lambda = (\hat{r}_2 - \hat{s}_{22})/\sqrt{N_0/2}$.

Therefore, in terms of the Q-function, the probability of error can be compactly expressed as:

$$\Pr[\text{error}] = P_1 Q\left(\frac{T-\hat{s}_{12}}{\sqrt{N_0/2}}\right) + P_2\left[1 - Q\left(\frac{T-\hat{s}_{22}}{\sqrt{N_0/2}}\right)\right]. \qquad (5.102)$$

Consider now the important case where the a priori probabilities are equal, i.e., $P_1 = P_2$. The threshold T becomes $T = (\hat{s}_{12} + \hat{s}_{22})/2$. Substituting this threshold into (5.102) and applying the property $Q(x) =$

$1-Q(-x)$ of the Q-function simplifies the expression of the error probability to

$$\Pr[\text{error}] = Q\left(\frac{\hat{s}_{22}-\hat{s}_{12}}{\sqrt{2N_0}}\right). \tag{5.103}$$

The term $(\hat{s}_{22}-\hat{s}_{12})$ can be further evaluated as follows:

$$\begin{aligned}
\hat{s}_{22}-\hat{s}_{12} &= \int_0^{T_b} s_2(t)\hat{\phi}_2(t)\mathrm{d}t - \int_0^{T_b} s_1(t)\hat{\phi}_2(t)\mathrm{d}t \\
&= \int_0^{T_b} [s_2(t)-s_1(t)]\hat{\phi}_2(t)\mathrm{d}t \\
&= \int_0^{T_b} [s_2(t)-s_1(t)]\frac{s_2(t)-s_1(t)}{(E_2-2\rho\sqrt{E_1}E_2+E_1)^{1/2}}\mathrm{d}t \\
&= \frac{\int_0^{T_b}[s_2(t)-s_1(t)]^2\mathrm{d}t}{(E_2-2\rho\sqrt{E_1E_2}+E_1)^{1/2}} = \frac{d_{21}^2}{d_{21}} = d_{21} \\
&= (E_2-2\rho\sqrt{E_1E_2}+E_1)^{1/2}.
\end{aligned} \tag{5.104}$$

The above result shows that $\hat{s}_{22}-\hat{s}_{12}$ simply measures the distance between the two signals $s_1(t)$ and $s_2(t)$ (see (5.21)), which is consistent with what is observed graphically in Figure 5.24-(b). The above evaluation also shows that, as long as $s_1(t) \neq s_2(t)$, the quantity $\hat{s}_{22}-\hat{s}_{12}$ is indeed positive. This is the assumption we used to arrive at the decision rule in (5.91).

Since the term $\sqrt{N_0/2}$ is interpreted as the RMS value of the noise at the output of the correlator or matched filter after sampling, one can write the expression of the error performance as

$$\Pr[\text{error}] = Q\left(\frac{\text{distance between the signals}}{2\times\text{noise RMS value}}\right) \tag{5.105}$$

$Q(\cdot)$ is a *monotonically decreasing* function of its argument, the probability of error decreases as the ratio $\left(\frac{\text{distance between the signals}}{2\times\text{noise RMS value}}\right)$ increases, i.e., as either the two signals become more dissimilar (increasing the distances between them) or the noise power becomes less. Both factors of course make it easier to distinguish between the two possible transmitted signals at the receiver.

Typically the channel noise power is fixed and thus the only way to reduce error is by maximizing the distance between the two signals. One way of doing this is of course by increasing the signal energy.

However the transmitter also has an energy constraint, say $\sqrt{E}$. Recall that in the signal space representation the distance from a signal point to the origin of the $\{\phi_1(1), \phi_2(t)\}$ plane is simply the square root of the signal energy. Thus the signals must lie on or inside a circle of radius $\sqrt{E}$. Therefore, to maximize the distance between the two signals one chooses them so that they are placed 180^0 from each other. This implies that $s_2(t) = -s_1(t)$ (you might want to show this). This very popular signal set is commonly known as *antipodal signalling*.

A last observation is that the error probability does *not* depend on the signal shapes but only on the distance between them. Vastly different signal sets (in terms of time waveforms) will lead to the same error peformance. This is due to the fact that our noise is considered to be white† and Gaussian.

Relationship between $Q(x)$ and erfc(x). Besides the Q-function, another function widely used in error probability calculation is the *complementary error function*, denoted by erfc$(\cdot)$. The erfc-function is defined as follows:

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} \exp(-\lambda^2)\text{d}\lambda \tag{5.106}$$

$$= 1 - \text{erf}(x). \tag{5.107}$$

By change of variables, it is not hard to show that the erfc-function and the Q-function are related by,

$$Q(x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right), \tag{5.108}$$

or conversely,

$$\text{erfc}(x) = 2Q(\sqrt{2}x). \tag{5.109}$$

Note that it is erfc(x), not $Q(x)$, that is available in MATLAB.

Finally, the next example is provided to illustrate some of the most important concepts introduced in this chapter.

Example 5.9. Consider the signal space diagram shown in Figure 5.32.

(a) Determine and sketch the two signals $s_1(t)$ and $s_2(t)$.

† "You can run but you cannot hide." Because white noise has equal power at each frequency one can say that the signals can run but cannot hide. The challenge to boxing fans is: What famous person uttered this phrase and in what circumstance?

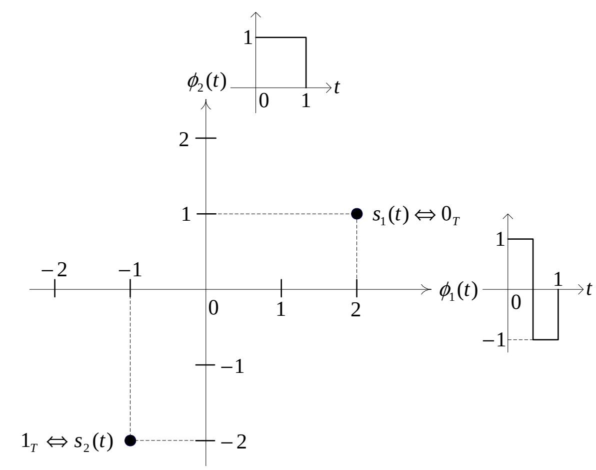


Fig. 5.32. Signal set for Example 5.9.

(b) The two signals $s_1(t)$ and $s_2(t)$ are used for the transmission of equally likely bits 0 and 1, respectively, over an additive white Gaussian noise (AWGN) channel. Clearly draw the decision boundary and the decision regions of the optimum receiver. Write the expression for the optimum decision rule.

(c) Find and sketch the two orthonormal basis functions $\hat{\phi}_1(t)$ and $\hat{\phi}_2(t)$ such that the optimum receiver can be implemented using only the projection $\hat{r}_2$ of the received signal $r(t)$ onto the basis function $\hat{\phi}_2(t)$. Draw the block diagram of such a receiver that uses a matched filter.

(d) Consider now the following argument put forth by your classmate. She reasons that since the component of the signals along $\hat{\phi}_1(t)$ is not useful at the receiver in determining which bit was transmitted, one should not even transmit this component of the signal. Thus she modifies the transmitted signal as follows:

$$s_1^M(t) = s_1(t) - \Big(\text{component of } s_1(t) \text{ along } \hat{\phi}_1(t)\Big) \quad (5.110)$$

$$s_2^M(t) = s_2(t) - \Big(\text{component of } s_2(t) \text{ along } \hat{\phi}_1(t)\Big) \quad (5.111)$$

Clearly identify the locations of $s_1^M(t)$ and $s_2^M(t)$ in the signal space diagram. What is the average energy of this signal set? Compare it to the average energy of the original set. Comment.

Solutions.

(a) The two signals $s_1(t)$ and $s_2(t)$ can be determined simply from their coordinates as:

$$s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t) = 2\phi_1(t) + \phi_2(t) \tag{5.112}$$
$$s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t) = -\phi_1(t) - 2\phi_2(t) \tag{5.113}$$

The two signals are plotted in Figure 5.33.

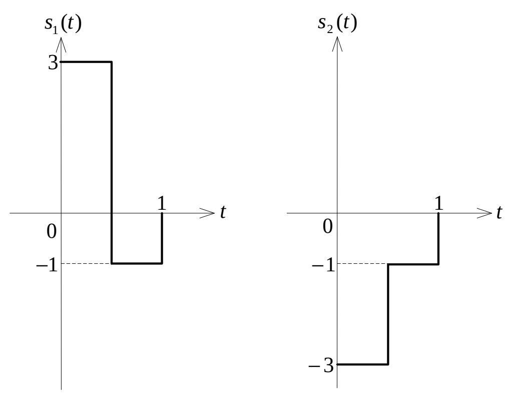


Fig. 5.33. Plots of $s_1(t)$ and $s_2(t)$.

(b) Since the two binary bits are equally likely, the decision boundary of the optimum receiver is the bisector of the line joining the two signals. The optimum decision boundary and the decision regions are shown in Figure 5.34. A simple inspection of the decision boundary in Figure 5.34 gives the following optimum decision rule:

$$r_1 \underset{1_D}{\overset{0_D}{\gtreqless}} -r_2. \tag{5.114}$$

Of course, the above expression for the optimum decision rule can be also be reached by substituting all the signal coordinates into the following fundamental minimum distance rule:

$$(r_1 - s_{21})^2 + (r_2 - s_{22})^2 \underset{\text{“}1_D\text{”}}{\overset{\text{“}0_D\text{”}}{\gtreqless}} (r_1 - s_{11})^2 + (r_2 - s_{12})^2. \tag{5.115}$$

(c) From the signal space diagram in Figure 5.34, it is clear that the orthonormal basis functions $\hat{\phi}_1(t)$ and $\hat{\phi}_2(t)$ are obtained by rotating

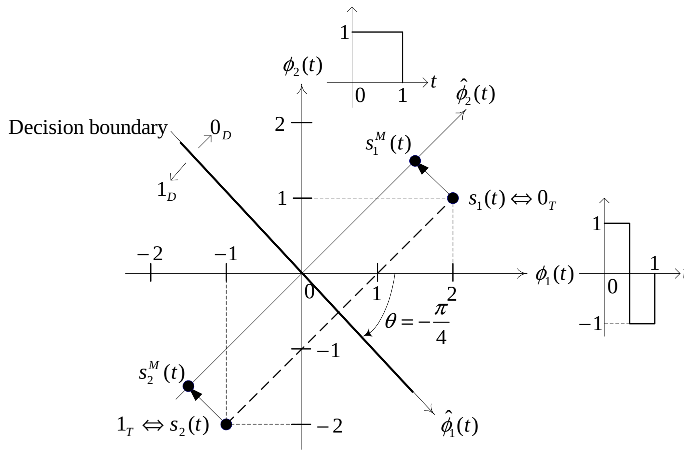


Fig. 5.34. Optimum decision boundary and decision regions.

$\phi_1(t)$ and $\phi_2(t)$ by 45^0 clockwise (i.e., $\theta = -\frac{\pi}{4}$), or 135^0 counterclockwise (i.e., $\theta = \frac{3\pi}{4}$). This rotation is to ensure that $\hat{\phi}_1(t)$ is perpendicular to the line joining the two signals. Choosing $\theta = -\frac{\pi}{4}$ yields:

$$\begin{bmatrix} \hat{\phi}_1(t) \\ \hat{\phi}_2(t) \end{bmatrix} = \begin{bmatrix} \cos(-\pi/4) & \sin(-\pi/4) \\ -\sin(-\pi/4) & \cos(-\pi/4) \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix}. \tag{5.116}$$

It follows that

$$\hat{\phi}_1(t) = \frac{1}{\sqrt{2}}[\phi_1(t) - \phi_2(t)] \tag{5.117}$$

$$\hat{\phi}_2(t) = \frac{1}{\sqrt{2}}[\phi_1(t) + \phi_2(t)] \tag{5.118}$$

These two functions are plotted in Figure 5.35 together with the block diagram of a receiver that uses one matched filter.

(d) The locations of $s_1^M(t)$ and $s_2^M(t)$ are shown on the signal space dia-

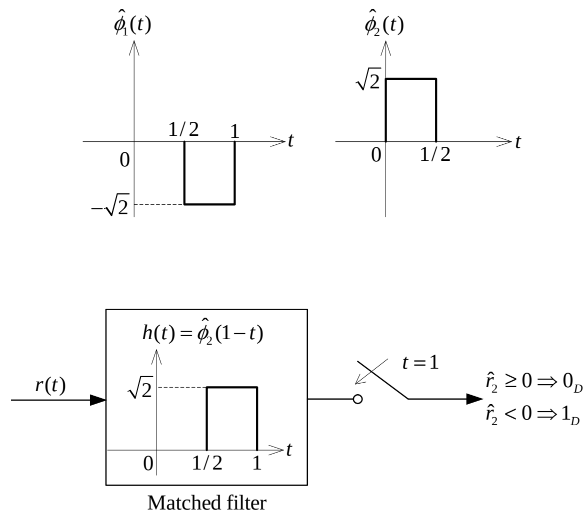


Fig. 5.35. Plots of $\hat{\phi}_1(t)$ and $\hat{\phi}_2(t)$ and the simplified receiver.

gram in Figure 5.34. The average energy of this signal set is simply:

$$\begin{aligned} E^M &= \frac{1}{2}(\hat{s}_{12}^2 + \hat{s}_{22}^2) = \frac{1}{2}[(E_1 - \hat{s}_{11}^2) + (E_2 - \hat{s}_{21}^2)] \\ &= \frac{1}{2}(E_1 + E_2) - \frac{1}{2}(\hat{s}_{11}^2 + \hat{s}_{21}^2) = 5 - \left(\frac{1}{\sqrt{2}}\right)^2 \\ &= 4.5 \text{ (joules).} \end{aligned} \tag{5.119}$$

The average energy of the modified signal set is clearly smaller than the average energy of the original set, which is $\frac{E_1+E_2}{2} = 5$ joules. Since the distance between the modified signals is the same as that of the original signals, both sets perform identically in terms of the bit error probability performance. The modified set is therefore preferred due to its better energy (or power) efficiency. It should be pointed out, however, that the other set might have better timing or spectral properties. Besides error performance, the issues of timing and bandwidth are also very important considerations and shall be explored further in the next chapter.

5.7 Summary

The foundation for the analysis, evaluation and design of a wide spectrum of digital communication systems has been developed in this chapter. Since the approach is so fundamental it is worthwhile to summarize the important features of it.

One starts with the modulator where a signal set comprised of two members, $s_1(t)$ and $s_2(t)$, is chosen to represent the binary digits "0" and "1" respectively. Then an orthonormal set, $\{\phi_1(t), \phi_2(t)\}$, is selected to represent the two signals exactly. This representation is expressed geometrically in a signal space plot. The received signal is corrupted by additive, white Gaussian noise (AWGN), $w(t)$, which is also represented by a series expansion using the orthonormal basis set, $\phi_1(t)$, $\phi_2(t)$, as dictated by, $s_1(t)$, $s_2(t)$, and the set $\phi_3(t)$, $\phi_4(t)$, $\ldots$, which are used to complete the set. However as shown the projections of the noise $w(t)$ onto the basis functions $\phi_3(t)$, $\phi_4(t)$, $\ldots$, do not provide any information as to which signal was transmitted, they are irrelevant statistics. Therefore these projections can be discarded or ignored which means that in practice the basis functions $\phi_3(t)$, $\phi_4(t)$,$\ldots$, do not need to be determined.

Having developed a representation of the transmitted signal, the additive noise and hence the received signal, $r(t)$, attention was turned to the development of the demodulator. To proceed the criterion of minimizing bit error probability was chosen. This resulted in the likelihood ratio test of Equation (5.65). One should observe that in the development of the likelihood ratio test no assumptions regarding the statistics of the received samples, r_1, r_2, r_3, $\ldots$, were made. The developed test is therefore quite general; what one needs to do is to determine the two conditional densities, $f(\mathbf{r}|1_T)$ and $f(\mathbf{r}|0_T)$.

Determination of these conditional densities is greatly simplified when the important channel model of additive white Gaussian noise is invoked. This is because the noise projections are Gaussian, uncorrelated and because they are *Gaussian and uncorrelated they are statistically independent*. Therefore as mentioned above the projections of the received signal onto $\phi_3(t)$, $\phi_4(t)$,$\ldots$, can be ignored. The receiver, in this situation, has a very intuitive interpretation either algebraically or graphically using the the signal space plot. Mainly it can be interpreted as a minimum distance receiver, i.e., choose the transmitter's signal to which the received signal is closest to, or as a maximum correlation receiver, i.e., choose the transmitter's signal with which the received signal is most correlated. This interpretation holds directly when the transmitter's two signals are equally probable, which is the typical situation. If not, then the distance or correlation needs to reflect this *a priori* knowledge.

The next two chapters apply the concepts developed in this chapter to important baseband and passband modulation schemes for binary digital communication systems. Though a formal approach known as Gram-Schmidt was presented in this chapter to determine the orthonormal basis $\{\phi_1(t), \phi_2(t)\}$ it shall be used sparingly, if at all, in the next two chapters. Indeed the basis set shall be determined by inspection. But to do this one should have a clear idea of and a firm grasp on what orthogonality is. Two signals, $s_1(t)$ and $s_2(t)$, are orthogonal over the interval of T seconds if and only if:

$$\int_T s_1(t)s_2(t)\mathrm{d}t = 0.$$

Problems

5.1 Clearly verify the final expression of $\phi_2(t)$ in (5.15) on page 79.

5.2 Assume that $s_1(t)$ and $s_1(t)$ are represented by an orthonormal basis set $\{\phi_1(t), \phi_2(t)\}$ as shown in Figure 5.36.

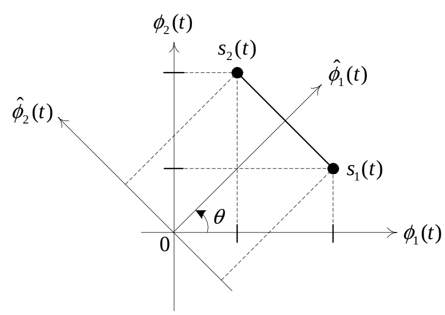


Fig. 5.36. Problem 5.2: Rotation of $\phi_1(t)$ and $\phi_2(t)$ by θ.

Now consider the rotation of $\phi_1(t)$ and $\phi_2(t)$ by an angle θ to obtain a new set of functions $\{\hat{\phi}_1(t), \hat{\phi}_2(t)\}$. This rotation can be expressed as,

$$\begin{pmatrix}\hat{\phi}_1(t)\\ \hat{\phi}_2(t)\end{pmatrix} = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{pmatrix}\phi_1(t)\\ \phi_2(t)\end{pmatrix}. \tag{P5.1}$$

(a) Show that, regardless of the angle θ, the set $\{\hat{\phi}_1(t), \hat{\phi}_2(t)\}$ is also an orthonormal basis set.

Remark: Note that the above result means that the projection of white noise of spectral strength $N_0/2$ (watts/Hz) onto the basis set $\{\hat{\phi}_1(t), \hat{\phi}_2(t)\}$ still results in zero-mean uncorrelated random variables with variance $N_0/2$ (watts).

(b) What are the values of θ that make $\hat{\phi}_1(t)$ perpendicular to the line joining $s_1(t)$ to $s_2(t)$? For these values of θ, mathematically show that the components of $s_1(t)$ and $s_2(t)$ along $\hat{\phi}_1(t)$, namely $\hat{s}_{11}$ and $\hat{s}_{21}$, are identical.

5.3 The Q-function is defined as the area under a zero-mean, unit-variance Gaussian curve from x to ∞. In determining the error performance of the receiver in digital communications, one often needs to compute the area from a threshold T to infinity under a general Gaussian density function with mean μ and variance σ^2 (see Figure 5.37). Show that such an area can be expressed as $Q\left(\frac{T-\mu}{\sigma}\right)$, $-\infty \le T \le \infty$.

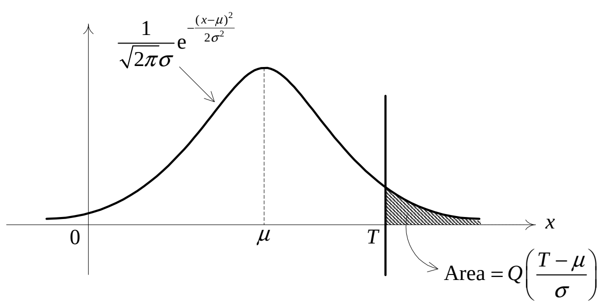


Fig. 5.37. Problem 5.3: The area under the right tail of a Gaussian pdf.

The next eight problems deal with the representation of signals using the signal space diagram. It is almost impossible to overestimate the importance of this concept in digital communications. The reader is strongly urged to attempt all problems.

5.4 (a) Consider two arbitrary signals $s_1(t)$ and $s_2(t)$ whose energies are E_1 and E_2, respectively. Both signals are time-limited over $0 \le t \le T_b$. It is known that two *orthonormal* functions $\phi_1(t)$ and $\phi_2(t)$ can be used to exactly represent $s_1(t)$ and

$s_2(t)$ as follows:

$$\begin{cases} s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t) \\ s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t) \end{cases} \quad \text{(P5.2)}$$

Show that $E_i = s_{i1}^2 + s_{i2}^2$ by directly evaluating $\int_0^{T_b} s_i^2(t)\mathrm{d}t$, $i = 1, 2$.

(b) Let

$$\phi_1(t) = \begin{cases} \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t), & 0 \le t \le T_b \\ 0, & \text{otherwise} \end{cases} \quad \text{(P5.3)}$$

and

$$\phi_2(t) = \begin{cases} \sqrt{\frac{2}{T_b}}\sin(2\pi f_c t), & 0 \le t \le T_b \\ 0, & \text{otherwise} \end{cases} \quad \text{(P5.4)}$$

Find the minimum value of frequency f_c that makes $\phi_1(t)$ and $\phi_2(t)$ orthogonal.

Remark: The signal set considered in (b) is an important one in passband communication systems, not only binary but also M-ary.

5.5 Consider the following two signals $s_1(t)$ and $s_2(t)$ that are time-limited to $[0, T_b]$:

$$s_1(t) = V\cos(2\pi f_c t) \quad \text{(P5.5)}$$

$$s_2(t) = V\cos(2\pi f_c t + \theta) \quad \text{(P5.6)}$$

where $f_c = \frac{k}{2T_b}$ and k is an integer.

(a) Find the energies of both signals. Then determine the value of V so that both signals have an unit energy.

(b) Determine the correlation coefficient ρ of the two signals. Recall that

$$\rho = \frac{1}{\sqrt{E_1 E_2}}\int_0^{T_b} s_1(t)s_2(t)\mathrm{d}t \quad \text{(P5.7)}$$

(c) Plot ρ as a function of θ over the range $0 \le \theta \le 2\pi$. What is the value of θ that makes the two signals orthogonal?

(d) Verify that the distance between the two signals is $d = \sqrt{2E}\sqrt{1-\rho}$. What is the value of θ that maximizes the distance between the two signals?

Remark: The above is another important signal set. Relate the value of θ found in c) to Problem 5.4-(b). Sketch the answers in (c) and (d) in the signal space to get geometrical insight. Finally to determine the energies in (a): you may do it directly or reason as follows. The RMS (root mean squared) of the signal is say V_{RMS} (volts) and, for a sinusoid with an integer number of cycles in the time interval T_b, should be known by you from an elementary signals course. The average dissipated power across a 1 ohm resistor is then $\frac{V_{\text{RMS}}^2}{1}$ (watts). Multiplying by the time interval gives the average energy of the signal $V_{\text{RMS}}^2 T_b$ (watts $\times$ sec $=$ joules).

5.6 *In contrast to Problem 5.5 where the phase of a sinusoid is used to distinguish between the two signals, here the frequency is used. Though the frequency can be chosen to maximize the distance between the signals, in practice the frequency separation is chosen so that the two signals are orthogonal. This greatly simplifies receiver design and also makes synchronization easier, as shall be seen in later chapters.*

Consider the following signal set over $0 \le t \le T_b$:

$$s_1(t) = V_1 \cos\left[2\pi\left(f_c - \frac{\Delta f}{2}\right)t\right] \tag{P5.8}$$

$$s_2(t) = V_2 \cos\left[2\pi\left(f_c + \frac{\Delta f}{2}\right)t\right] \tag{P5.9}$$

where $f_c = \frac{k}{2T_b}$ and k is an integer. The amplitudes V_1 and V_2 are adjusted so that regardless of Δf the energies E_1, E_2 are always the same and equal to $E = \frac{V^2 T_b}{2}$ joules.

(a) Determine and plot the correlation coefficient, ρ, as a function of Δf.

(b) Given that the distance between two equal-energy signals is $d = \sqrt{2E}\sqrt{1-\rho}$, show that the distance between the two above signals is maximum when $\Delta f = \frac{0.715}{T_b}$. Compute the distance between $s_1(t)$ and $s_2(t)$ for this Δf. How much has the distance increased as compared to the case $\rho = 0$.

5.7 *This problem results in an orthonormal set that are a subset of the well known Legendre polynomials. As such they are not particularly important in communications. The problem is included to strengthen your understanding of the Gram-Schmidt procedure.*

Using the Gram-Schmidt procedure, construct an orthonormal basis for the space of quadratic polynomials $\{a_2t^2 + a_1t + a_0;\ a_0, a_1, a_2 \in \mathbb{R}\}$ over the interval $-1 \le t \le 1$.

Hint: The equivalent problem is to find an orthonormal basis for three signals $\{1, t, t^2\}$ over the interval $-1 \le t \le 1$.

5.8 Consider a set of M orthogonal signal waveforms $s_m(t)$, $1 \le m \le M$, $0 \le t \le T$, all of which have the same energy E. Define a new set of M waveforms as

$$\hat{s}_m(t) = s_m(t) - \frac{1}{M}\sum_{k=1}^{M} s_k(t), \quad 1 \le m \le M, \quad 0 \le t \le T. \tag{P5.10}$$

Show that the M signal waveforms $\{\hat{s}_m(t)\}$ have equal energy, given by

$$\hat{E} = (M-1)E/M \tag{P5.11}$$

and are equally correlated, with correlation coefficient

$$\rho_{mn} = \frac{1}{\hat{E}}\int_0^T \hat{s}_m(t)\hat{s}_n(t)\mathrm{d}t = -\frac{1}{M-1} \tag{P5.12}$$

Remark: The signal set obtained here is the well known ***simplex set****. Signal space plots of them for $M = 2, 3, 4$ are quite informative.*

5.9 (*A generalization of the Fourier approach to approximate an energy signal*) Suppose that $s(t)$ is a deterministic, real-valued signal with finite energy $E_s = \int_{-\infty}^{\infty} s^2(t)\mathrm{d}t$. Furthermore suppose that there exists a set of orthonormal basis functions $\{\phi_n(t), n = 1, 2, \ldots, N\}$, i.e.,

$$\int_{-\infty}^{\infty} \phi_n(t)\phi_m(t)\mathrm{d}t = \begin{cases} 0, & (m \ne n) \\ 1, & (m = n) \end{cases} \tag{P5.13}$$

We want to approximate the signal $s(t)$ by a weighted linear combination of these basis functions, i.e.,

$$\hat{s}(t) = \sum_{k=1}^{N} s_k\phi_k(t) \tag{P5.14}$$

where $\{s_k, k = 1, 2, \ldots, N\}$ are the coefficients in the approximation of $s(t)$. The approximation error incurred is

$$e(t) = s(t) - \hat{s}(t) \tag{P5.15}$$

(a) Find the coefficients $\{s_k\}$ to minimize the energy of the approximation error.

(b) What is the minimum mean square approximation error, i.e., $\int_{-\infty}^{\infty} e^2(t)\mathrm{d}t$?

5.10 (*This problem emphasizes the geometrical approach to signal representation*) Consider two signals $s_1(t)$ and $s_2(t)$ as plotted in Figure 5.38-(b). The two orthonormal basis functions $\phi_1(t)$ and $\phi_2(t)$ in Figure 5.38-(a) are chosen to represent the two signals $s_1(t)$ and $s_2(t)$, i.e.,

$$\begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} \tag{P5.16}$$

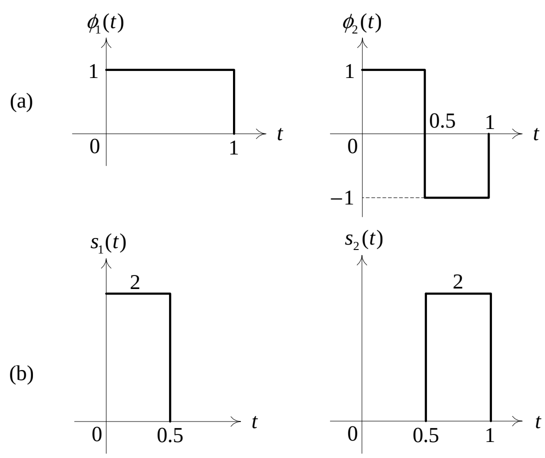


Fig. 5.38. Problem 5.10: Orthonormal functions and signal set.

(a) Determine the coefficients s_{ij}, $i, j \in \{1, 2\}$.

(b) Consider a new set of orthonormal functions $\phi_1^R(t)$ and $\phi_2^R(t)$, which are $\phi_1(t)$ and $\phi_2(t)$ axes rotated by an angle of θ degree, i.e.,

$$\begin{bmatrix} \phi_1^R(t) \\ \phi_2^R(t) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} \tag{P5.17}$$

First, analytically show that $\phi_1^R(t)$ and $\phi_2^R(t)$ are indeed two orthonormal functions for any θ. Then determine and draw the *time waveforms* of the new orthonormal functions for $\theta = 60^0$.

(c) Determine the new set of coefficients s_{ij}^R, $i, j \in \{1, 2\}$ in the representation:

$$\begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} = \begin{bmatrix} s_{11}^R & s_{12}^R \\ s_{21}^R & s_{22}^R \end{bmatrix} \begin{bmatrix} \phi_1^R(t) \\ \phi_2^R(t) \end{bmatrix} \tag{P5.18}$$

(d) Plot the geometrical picture for both basis function sets (original and rotated) and also for the two signal points.

(e) Determine the distance d between the two signals $s_1(t)$ and $s_2(t)$ in two ways:

(i) Algebraically: $d = \sqrt{\int_0^1 [s_1(t) - s_2(t)]^2 \, dt}$.

(ii) Geometrically: From the signal space plot of (d) above.

(f) Thought $\phi_1(t)$ and $\phi_2(t)$ can represent the two given signals, they are by no means a complete basis set because they cannot represent an arbitrary, finite-energy signal defined on the time interval $[0, 1]$. As a start to complete the basis, plot the next two possible orthonormal functions $\phi_3(t)$ and $\phi_4(t)$ of the basis set.

5.11 (*Again more practice in determining an orthonormal basis set to represent the signal set exactly but in the M-ary, $M = 4$, case*) Consider the set of four time-limited waveforms shown in Fig. 5.39.

(a) Using the Gram-Schmidt procedure, construct a set of orthonormal basis functions for these waveforms.

(b) By inspection, show that the set of orthonormal functions in Fig. 5.40 can also be used to exactly represent the four signals in Fig. 5.39.

(c) Clearly plot the geometrical representation of the set of four signals $\{s_1(t), s_2(t), s_3(t), s_4(t)\}$ in the three-dimensional signal space spanned by $\{g_1(t), g_2(t), g_3(t)\}$.

The next three problems consider aspects of the design of a minimum error probability communication system.

5.12 (*An orthogonal binary signal set*) Consider the following signal set for binary data transmission over a channel disturbed by

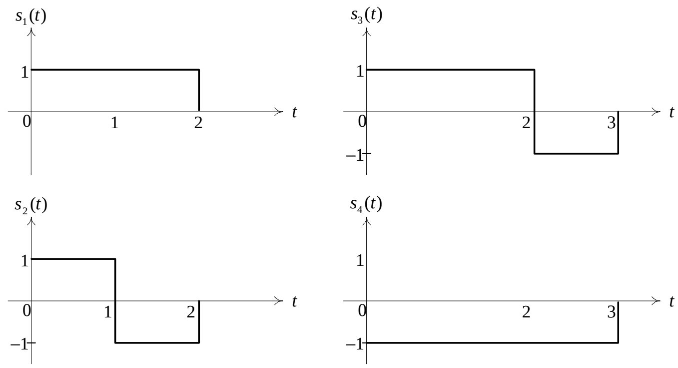


Fig. 5.39. Problem 5.11: A set of four time-limited waveforms.

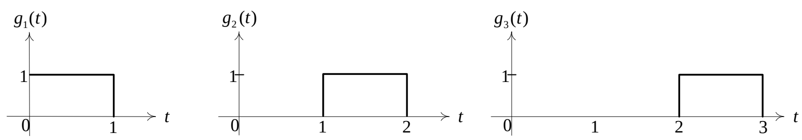


Fig. 5.40. Problem 5.11: A set of three orthonormal functions.

additive white Gaussian noise:

$$s_1(t) = \begin{cases} A\cos\left(\frac{2\pi t}{T_b}\right), & 0 \le t \le T_b \\ 0, & \text{otherwise} \end{cases} \quad \text{(P5.19)}$$

$$s_2(t) = \begin{cases} \sqrt{3}A\sin\left(\frac{2\pi t}{T_b}\right), & 0 \le t \le T_b \\ 0, & \text{otherwise} \end{cases} \quad \text{(P5.20)}$$

The noise is zero-mean and has two-sided power spectral density $N_0/2$. As usual, $s_1(t)$ is used for the transmission of bit "0" and $s_2(t)$ is for the transmission of bit "1". Furthermore, the two bits are equally likely.

(a) Show that $s_1(t)$ is *orthogonal* to $s_2(t)$. Then find and draw an orthonormal basis $\{\phi_1(t),\phi_2(t)\}$ for the signal set.

(b) Draw the signal space diagram and the optimum decision regions. Write the expression for the optimum decision rule.

(c) Let $A = 1V$ and assume that $N_0 = 10^{-8}$ watts/Hz. What is the maximum bit rate that can be sent with a probability of error $\Pr[\text{error}] \le 10^{-6}$.

(d) Draw the block diagram of an optimum receiver that uses only one matched filter. Give the precise expression for the impulse response of the matched filter.

(e) Assume that, as long as the average energy of the signal set $\{s_1(t), s_2(t)\}$ stays the same, you can freely change (or move) both $s_1(t)$ and $s_2(t)$ in the same signal space. Modify them so that the probability of error is as small as possible. Explain your answer.

5.13 Consider the signal set in Fig. 5.41 for binary data transmission over an AWGN channel. The noise is zero-mean and has two-sided power spectral density $N_0/2$. As usual, $s_1(t)$ and $s_2(t)$ are used for the transmission of equally likely bits "0" and "1", respectively. Furthermore, you are told that the energy of $s_1(t)$ is $V^2T_b/3$.

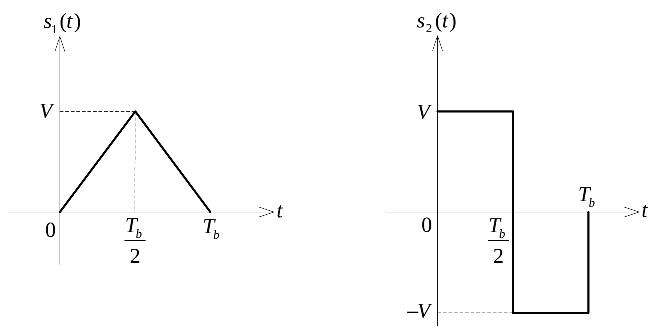


Fig. 5.41. Problem 5.13: A binary signal set.

(a) Show that $s_1(t)$ is *orthogonal* to $s_2(t)$. Then find and draw an orthonormal basis set $\{\phi_1(t),\phi_2(t)\}$ for the signal set.

(b) Draw the signal space diagram and the optimum decision regions. Write the expression for the optimum decision rule.

(c) Let $V = 1$ volt and assume that $N_0 = 10^{-8}$ watts/Hz. What is the maximum bit rate that can be sent with a probability of error $\Pr[\text{error}] \leq 10^{-6}$.

(d) Draw the block diagram of an optimum receiver that uses only one matched filter and clearly sketch the impulse response of the matched filter.

(e) Assume that the signal $s_1(t)$ is fixed. You however can change the shape, but not the energy, of $s_2(t)$. Modify $s_2(t)$ so that the probability of error is as small as possible. Explain your answer.

5.14 Consider the signal set in Figure 5.42 for binary data transmission over a channel disturbed by additive white Gaussian noise. The noise is zero-mean and has two-sided power spectral density $N_0/2$. As usual, $s_1(t)$ is used for the transmission of bit "0" and $s_2(t)$ is for the transmission of bit "1". Furthermore, the two bits are equiprobable.

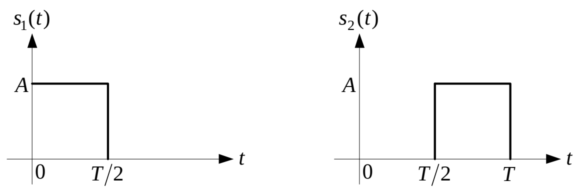


Fig. 5.42. Problem 5.14: A binary signal set.

(a) Find and draw an orthonormal basis $\{\phi_1(t),\phi_2(t)\}$ for the signal set.
(b) Draw the signal space diagram and the optimum decision regions. Write the expression for the optimum decision rule.
(c) Draw the block diagram of the receiver that implements the optimum decision rule in (b).
(d) Let $A = 1V$ and assume that $N_0 = 10^{-3}$ watts/Hz. What is the maximum bit rate that can be sent with a probability of error $\Pr[\text{error}] \leq 10^{-6}$.
(e) Draw the block diagram of an optimum receiver that uses only one correlator or one matched filter.
(f) Assume that signal $s_1(t)$ is fixed as above. You however can change $s_2(t)$. Modify it so that the average energy is maintained at $A^2T/2$ but the probability of error is as small as possible. Explain your answer.

5.15 (*An example of colored noise*) For a binary communication system with additive *white* Gaussian noise, the error performance of the optimum receiver does not depend on the specific signal shapes but simply on the distance between the two signals. This problem considers noise that is Gaussian but *not* white (i.e., it is *colored* noise). The communication system under consideration is shown in Figure 5.43.

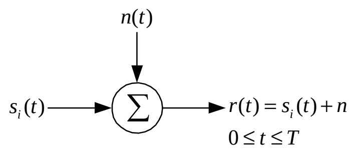


Fig. 5.43. Problem 5.15: A binary communication system with additive noise.

The noise, $n(t) = n$, is simply a DC level but the amplitude of the level is random and is Gaussian distributed with a probability density function given by

$$f(n) = \frac{1}{\sqrt{2\pi N_0}} \exp\left(-\frac{n^2}{2N_0}\right) \tag{P5.21}$$

(a) Determine the auto-correlation and the power spectral density of the noise. Are the successive noise samples correlated?

(b) Consider the signal set and the receiver shown in Figure 5.44. As usual, $s_1(t)$ is used for the transmission of bit "0" and $s_2(t)$ is for the transmission of bit "1". What is the error probability of this receiver?

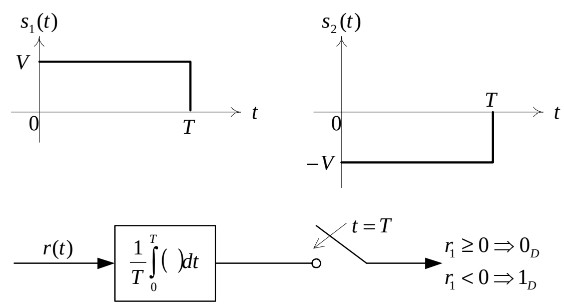


Fig. 5.44. Problem 5.15: The signal set and receiver considered in Part (b).

(c) Next consider the signal set in Figure 5.45. Find a receiver that will have an error probability of **zero**.

5.16 *In the development of the optimum demodulator the concept of the matched filter arose: either matched to the orthonormal basis set that is used to represent the signal set or directly to the signal set. The optimality criterion was minimum error probability. This problem shows that the matched filter has another "nice"*

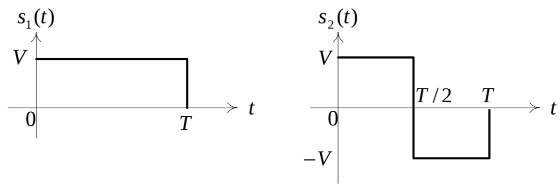


Fig. 5.45. Problem 5.15: The signal set considered in Part (c).

property, namely, it maximizes the signal to noise ratio at the output.

Consider the block diagram shown below:

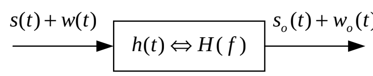


Fig. 5.46. Problem 5.16: A block diagram of the matched filter.

The signal $s(t)$ is known, i.e., deterministic, and duration limited to $[0, T_b]$ seconds. The input noise $w(t)$ is *white* with spectral strength of $N_0/2$. It is desired to determine the impulse response $h(t)$, or equivalently the transfer function $H(f)$, that *maximizes* the *signal-to-noise ratio* at the output at time $t = t_0$. The problem is therefore, choose $h(t)$ so that

$$\text{SNR}_{\text{out}} \equiv \frac{s_o^2(t_0)}{\mathcal{E}\{w_0^2(t)\}} = \frac{s_o^2(t_0)}{P_{w_o}} \qquad \text{(P5.22)}$$

is maximized. To find the solution, proceed as follows:

(a) Write the expression for $s_o(t)$ in terms of $H(f)$ and $S(f)$. From this get the expression for $s_o^2(t_0)$.

(b) Write the expression for the output noise power P_{w_o} (watts) in terms of $H(f)$ and the input noise power spectral density. Based on the results in (a) and (b) write the expression for SNR_{out}.

To proceed further we need the Schwarz inequality, which states that for any two *real* functions $a(t)$ and $b(t)$ one has:

$$\left[\int_{-\infty}^{\infty} a(t)b(t)\mathrm{d}t\right]^2 \leq \int_{-\infty}^{\infty} a^2(t)\mathrm{d}t \int_{-\infty}^{\infty} b^2(t)\mathrm{d}t \qquad \text{(P5.23)}$$

(c) Prove this inequality. The starting point is to consider $\int_{-\infty}^{\infty}[a(t)+\lambda b(t)]^2 \mathrm{d}t \geq 0$ (always). Expand the square and observe that the quadratic in λ cannot have any real roots to arrive at the inequality. When does the equality hold?

(d) Recall that $\int_{-\infty}^{\infty} a(t)b(t)\mathrm{d}t = \int_{-\infty}^{\infty} A(f)B^*(f)\mathrm{d}f$. (Why?) Rewrite the Schwarz inequality in the frequency domain, i.e., in terms of $A(f)$ and $B(f)$.

(e) Identify: $A(f) = H(f)$ and $B^*(f) = S(f)\mathrm{e}^{j2\pi f t_0}$. Argue that $\mathrm{SNR}_{\mathrm{out}} \leq \int_{-\infty}^{\infty} \frac{|S(f)|^2}{N_0/2}\mathrm{d}f$.

(f) Choose $H(f)$ to achieve equality and to complete the problem, find the corresponding $h(t)$.

5.17 A communication magazine carries an advertisement that advertises bargain basement pieces on a filter that has the following impulse response.

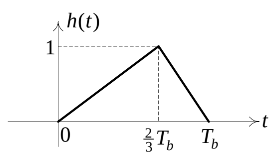


Fig. 5.47. Problem 5.17: An impulse response.

The manager of your division notices this and asks you to design an antipodal binary modulator for an AWGN channel with two-sided power spectral density of $N_0/2$ (watts/Hz). The goal of the design is to take advantage of this filter in the receiver for an optimum demodulation.

(a) Please design the modulator, basically the signal set to take advantage of this filter. Explain.

(b) Assume that $N_0 = 10^{-4}$ watts/Hz. What is the maximum bit rate that can be sent with a probability of error $\Pr[\text{error}] \leq 10^{-3}$.

Remark: In doing this problem a hint is that one can match the filter to the signal or match the signal to the filter. A paraphrase of the biblical statement: one can either bring the mountain to Moses or Moses to the mountain.

5.18 (*Antipodal signalling*) The received signal in a binary communication system that employs antipodal signals is

$$r(t) = s_i(t) + n(t) = \begin{cases} s(t) + n(t) & \text{if "0" is transmitted} \\ -s(t) + n(t) & \text{if "1" is transmitted} \end{cases} \quad \text{(P5.24)}$$

where $s(t)$ is shown in Figure 5.48 and $n(t)$ is AWGN with power-spectral density $N_0/2$ (watts/Hz).

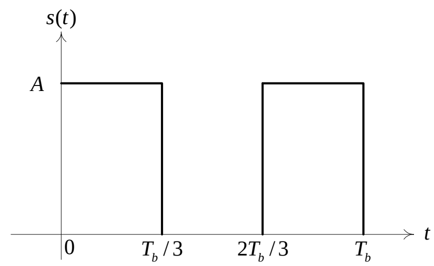


Fig. 5.48. Problem 5.18: Plot of $s(t)$.

(a) Sketch the impulse response of the filter *matched* to $s(t)$.
(b) Precisely sketch the output of the filter **to the input** $s(t)$ (i.e., "0" was transmitted and noise is ignored).
(c) Precisely sketch the output of the filter **to the input** $-s(t)$ (i.e., "1" was transmitted and noise is ignored).
(d) Compute the signal-to-noise ratio (SNR) at the output of the the filter at $t = T_b$.
(e) Determine the probability of error as a function of SNR.
(f) Plot the probability of error (in a log scale) as a function of SNR (in dB). What is the minimum SNR to achieve a probability of error of 10^{-6}?

5.19 (*Bandwidth*) Consider an antipodal binary communication system, where the two signals $s(t)$ and $-s(t)$ are used to transmit bits 0 and 1 respectively in every T_b seconds. You have learned in this chapter that the error performance of the optimum receiver for such a system depends on the energy, NOT the specific shape of the signal $s(t)$. Despite this fact, the choice of $s(t)$ is important in any practical design because it determines the transmission bandwidth of the system.

Since the signal $s(t)$ is time-limited to T_b seconds, it cannot be band-limited and a bandwidth definition is required. Here we define W to be the bandwidth of the signal $s(t)$ if $\epsilon\%$ of the total energy of $s(t)$ is contained inside the band $[-W, W]$. Mathematically, this means that:

$$\frac{\int_{-W}^{W} |S(f)|^2 \mathrm{d}f}{\int_{-\infty}^{\infty} |S(f)|^2 \mathrm{d}f} = \frac{2\int_{0}^{W} |S(f)|^2 \mathrm{d}f}{E} = \frac{\epsilon}{100} \qquad \text{(P5.25)}$$

where $S(f)$ is the Fourier transform of $s(t)$. Consider the following three signals:

(i) Rectangular pulse: $s(t) = \sqrt{\frac{1}{T_b}}$, $0 \le t \le T_b$.

(ii) Half-sine: $s(t) = \sqrt{\frac{2}{T_b}} \sin\left(\frac{\pi t}{T_b}\right)$, $0 \le t \le T_b$.

(iii) Raised cosine: $s(t) = \sqrt{\frac{2}{3T_b}} \left[1 - \cos\left(\frac{2\pi t}{T_b}\right)\right]$, $0 \le t \le T_b$.

Note that all the three signals have been normalized to have unit energy, i.e., $E = 1$ (joules).

(a) Derive (P5.25) for the above three signals. Try to put the final expressions in the form such that WT_b appears only in the limit of the integrals.

(b) Based on the expressions obtained in (a), evaluate WT_b for the three signals for each of the following values of ϵ: $\epsilon = 90$, $\epsilon = 95$ and $\epsilon = 99$.

Note: For this part, you need to do the integration numerically. In MATLAB, the routine `quadl` is useful for numerical integration. Type `help quadl` to see how to use this routine.

5.20 In antipodal signalling, two signals $s(t)$ and $-s(t)$ are used to transmit equally likely bits 0 and 1, respectively. Consider two communication systems, called system-(i) and system-(ii), that use two different time-limited waveforms as shown in Figure 5.49.

(a) What is the relationship between the parameters A and B of the two waveforms if the two communication systems have the same error performance? Explain your answer.

(b) Since the two signals in Figure 5.49 are time-limited, they cannot be band-limited and a bandwidth definition is required. Here we define W to be the bandwidth of the signal $s(t)$ if 95% of the total energy of $s(t)$ is contained inside the band $[-W, W]$. Assume that both systems have the same bit rate

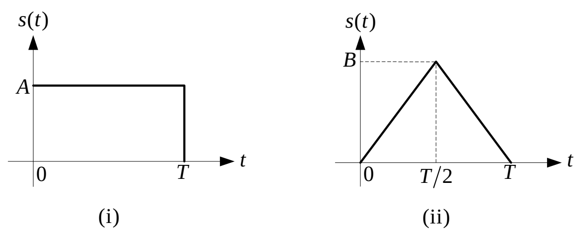


Fig. 5.49. Problem 5.20: Two possible waveforms for antipodal signalling.

of 2 Mbps. What are the required bandwidths of the two systems? What system is preferred and why?

Hint: The bandwidths of the above two signals have been found to be $W_1 = 2.07/T$ and $W_2 = 1.00/T$.

(c) Consider the system that uses $s(t)$ in Figure 5.49-(i). How large does the voltage level A need to be to achieve an error probability of 10^{-6} if the bit rate is 2 Mbps and $\frac{N_0}{2} = 10^{-8}$ (watts/Hz)?

5.21 (*A diversity system*) Consider an antipodal signalling system in Figure 5.50, where the signal $s(t)$ and $-s(t)$ are used to transmit the information bits "1" and "0" respectively. The bit duration is T and $s(t)$ is assumed to have unit energy. The signal is sent via two different channels, denoted 'A' and 'B', to the same destination. Each channel is described by a gain factor (V_A or V_B) and additive white Gaussian noise ($w_A(t)$ or $w_B(t)$). The noises $w_A(t)$ and $w_B(t)$ both have zero means and power spectral density of σ_A^2 and σ_B^2 respectively. Furthermore they are independent noise sources.

The receiver for such a system consists of two correlators, one for each channel, as shown in Figure 5.50. The output of one correlator, say the one corresponding to channel A, is passed through an amplifier with an adjustable voltage gain K. The signals are then added before being compared with a threshold of zero to make the decision.

(a) For a fixed K, find the probability of error of this system. *Hint*: The noise samples w_A and w_B are independent, zero mean, Gaussian random variables with variances σ_A^2 and σ_B^2 respectively. Furthermore, the sum of two independent Gaus-

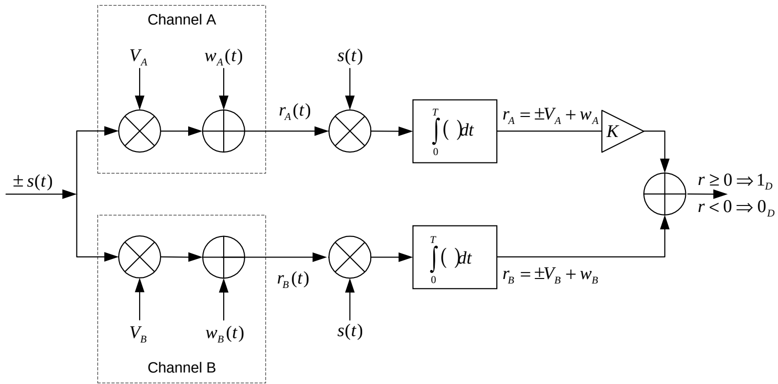


Fig. 5.50. Problem 5.21: A diversity system.

sian random variables is a Gaussian random variable whose variance equals the sum of the individual variances.

(b) Find the value of K that minimizes the probability of error. *Hint*: Minimizing $Q(x)$ is equivalent to maximizing x.

(c) What is the probability of error when the optimum value of K is used?

(d) What is the probability of error when K is simply set to 1?

Remark: Diversity is a modulation technique that is used to combat fading, a channel degradation commonly experienced in wireless communications.

The next two problems look at channels that are not modelled as AWGN. Though the general results obtained for the AWGN channel do not apply directly one can still use the concepts developed in this chapter. In particular, as mentioned in the summary section, the likelihood ratio test of Equation (5.65) holds.

5.22 (*Detection in Laplacian noise*) Consider the communication system model in Figure 5.51, where $Pr[s_1(t)] = P_1$ and $Pr[s_2(t)] = P_2$. The noise is modeled to be Laplacian. At the receiver, you sample $r(t)$ uniformly m times within the time period $[0, T_b]$. The samples are taken far enough apart so that you feel reasonably confident that the noise samples $r_j = r(jT_b/m)$, $j = 1, \ldots, m$, are statistically independent.

(a) Determine the two conditional density functions for a single sample, i.e., $f(r_j|1_T)$ and $f(r_j|0_T)$. From this what are the

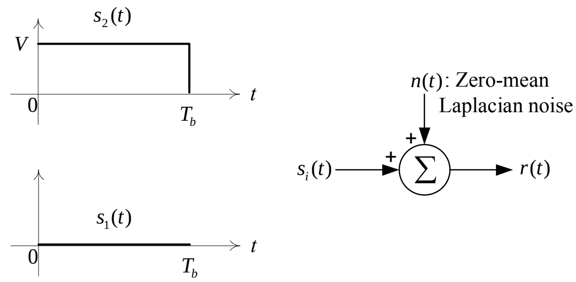


Fig. 5.51. Problem 5.22: A binary communication system with additive noise.

conditional density functions $f(r_1, r_2, \cdots, r_m|1_T)$ and $f(r_1, r_2, \cdots, r_m|0_T)$?

(b) Find the ln of the likelihood ratio:

$$\frac{f(r_1, r_2, \cdots, r_m|1_T)}{f(r_1, r_2, \cdots, r_m|0_T)} \tag{P5.26}$$

Simplify the sum as much as possible by considering the samples that fall in three different ranges: $r_j < 0$ with m_1 samples; $0 < r_j < V$ with m_2 samples; $V < r_j$ with m_3 samples, where $m_1 + m_2 + m_3 = m$. Considering the three different ranges should allow you to eliminate the magnitude operation.

(c) Derive the decision rule that minimizes the error probability.

5.23 (*Optical communications*) As one goes higher and higher in frequency, the wavelike nature of electromagnetic radiation recedes into the background and quantum effects become more pronounced. In optical communications, one turns on the source (say a semiconductor laser) for a fixed interval of time and lets it radiate energy (photons). This represents the binary digit "1". To transmit a "0" the laser is switched off but because of background radiation, there are photons arriving at the receiver even when the laser is switched off. A common model for the number of photons emmited per unit time is that it is random and behaves like a Poisson point process. Thus

$$\Pr[k \text{ photons emitted in a unit interval}] = \frac{\lambda^k e^{-\lambda}}{k!},\ k = 0, 1, \ldots, \tag{P5.27}$$

where λ is the mean arrival rate with units of photons per unit time. Let the signalling interval be T_b seconds. The receiver would count the number of photons received in the interval $[0, T_b]$ and based on this count make a decision as to whether a "0" or "1" was transmitted. Assume that the probability of transmitting a "1" is the same as transmitting "0". Note also that $\lambda = \lambda_s + \lambda_n$ or $\lambda = \lambda_n$, where λ_s is due to the transmitter laser being turned on and λ_n is due to the background radiation.

(a) Design the receiver that minimizes the probability of error.

(b) After designing the receiver, derive an expression for the error performance. Note that the average transmitted energy is hfT_b, where hf is the energy of photon with frequency f, h is Planck's constant ($h = 6.6 \times 10^{-34}$) and $\lambda_s T_b$ is the average number of signal photons per bit interval. Is the error performance dependent on "some" SNR?

5.24 (*Signal design for non-uniform sources*) Consider a binary communication system where $s_1(t)$ and $s_2(t)$ are used for the transmission of bits "0" and "1", respectively, over an AWGN channel. The bit duration is T_b and both $s_1(t)$ and $s_2(t)$ are time-limited to $[0, T_b]$. Furthermore, the a priori probabilities of the information bits are:

$$\Pr[0_T] = \Pr[s_1(t)] = p \le 0.5,\ \Pr[1_T] = \Pr[s_2(t)] = 1 - p \ge 0.5 \tag{P5.28}$$

The two signals $s_1(t)$ and $s_2(t)$ have energies E_1 and E_2. The optimum receiver for such a system implements the following decision rule:

$$\int_0^{T_b} r(t)[s_2(t) - s_1(t)]\mathrm{d}t \underset{0_D}{\overset{1_D}{\gtreqless}} \frac{E_2 - E_1}{2} + \frac{N_0}{2}\ln\left(\frac{p}{1-p}\right) \tag{P5.29}$$

where $r(t)$ is the received signal and $\frac{N_0}{2}$ is the power spectral density of the AWGN.

It was derived in this chapter that the error performance of the above optimum receiver is:

$$\Pr[\text{error}] = pQ\left(\frac{T - \hat{s}_{12}}{\sqrt{N_0/2}}\right) + (1-p)\left[1 - Q\left(\frac{T - \hat{s}_{22}}{\sqrt{N_0/2}}\right)\right] \tag{P5.30}$$

where

$$T = \frac{\hat{s}_{12} + \hat{s}_{22}}{2} + \frac{N_0}{2(\hat{s}_{22} - \hat{s}_{12})}\ln\left(\frac{p}{1-p}\right) \tag{P5.31}$$

and $\hat{s}_{12}$, $\hat{s}_{22}$ are the projections of $s_1(t)$ and $s_2(t)$ onto $\hat{\phi}_2(t)$, respectively. The basis function $\hat{\phi}_2(t)$ is given by:

$$\hat{\phi}_2(t) = \frac{s_2(t) - s_1(t)}{(E_2 - 2\rho\sqrt{E_1 E_2} + E_1)^{1/2}}, \tag{P5.32}$$

with

$$\rho = \frac{1}{\sqrt{E_1 E_2}} \int_0^{T_b} s_1(t) s_2(t) \mathrm{d}t. \tag{P5.33}$$

The above results apply for an arbitrary source distribution (i.e., arbitrary value of $p \leq 0.5$) and arbitrary signal set $\{s_1(t), s_2(t)\}$. Now consider the case that p is fixed but one can freely design $s_1(t)$ and $s_2(t)$ to minimize Pr[error] in (P5.30). Of course, the design is subject to a constraint on the average transmitted energy:

$$\bar{E}_b = E_1 p + E_2(1-p) \tag{P5.34}$$

(a) Show that the error probability in (P5.30) can be written as:

$$\Pr[\text{error}] = pQ\left(\sqrt{A} - \frac{B}{\sqrt{A}}\right) + (1-p)Q\left(\sqrt{A} + \frac{B}{\sqrt{A}}\right) \tag{P5.35}$$

where $B = 0.5\ln\left(\frac{1-p}{p}\right) \geq 0$ and $A = \frac{(\hat{s}_{22}-\hat{s}_{12})^2}{2N_0} = \frac{d_{21}^2}{2N_0} = \frac{E_2 - 2\rho\sqrt{E_1E_2}+E_1}{2N_0}$. It can be shown from (P5.35) that Pr[error] is minimized when A, or equivalently d_{21}, is *maximized*.

(b) Consider a special case of orthogonal signalling, i.e., $\rho = 0$. Design the two signals $s_1(t)$ and $s_2(t)$ to minimize Pr[error].

(c) Assume that $p = 0.1$. Use MATLAB to plot on the same figure the Pr[error] of the system that uses your signal design obtained in (b) and also the Pr[error] of the system that uses the "conventional" design of $E_1 = E_2 = \bar{E}_b$. Show your plots over the ranges [0 : 1 : 20] dB for $\bar{E}_b/N_0$ and $[10^{-7} \rightarrow 10^0]$ for Pr[error]. What is the gain in $\bar{E}_b/N_0$ of your design over the conventional design at $\Pr[\text{error}] = 10^{-5}$?

(d) Now let ρ be arbitrary. Design the two signals $s_1(t)$ and $s_2(t)$ to minimize Pr[error]. Repeat (c) for the signal set obtained in this part and antipodal signalling (also assume that $p = 0.1$). *Hint:* Argue that the optimal signal set must correspond to $\rho < 0$.

5.25 A ternary communication system transmits one of three signals, $s(t)$, 0, or $-s(t)$ every T seconds. The received signals is either $r(t) = s(t) + w(t)$, $r(t) = w(t)$ or $r(t) = -s(t) + w(t)$ where $w(t)$ is white Gaussian noise with zero mean and power spectral density of $N_0/2$. The optimum receiver computes the correlation metric

$$U = \int_0^T r(t)s(t)\mathrm{d}t \tag{P5.36}$$

and compares U with a threshold A and a threshold $-A$. If $U > A$, the decision is made that $s(t)$ was sent. If $U < -A$ the decision is made in favor of $-s(t)$. If $-A \leq U \leq A$, the decision is made in favor of 0.

(a) Determine the three conditional probabilities of error: P_e given that $s(t)$ was sent, P_e given that $-s(t)$ was sent, and P_e given that 0 was sent.

(b) Determine the average probability of error P_e as a function of the threshold A, where the a priori probabilities of the three signals are $\Pr[s(t)] = \Pr[-s(t)] = \frac{1}{4}$ and $\Pr[0] = \frac{1}{2}$.

(c) Determine the value of A that minimizes the average probability of error when the a priori probabilities of three signals are given as in (c). How does the value of A change if three signals are equally probable?

Hint:

$$\frac{\partial Q(x)}{\partial x} = -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{P5.37}$$

Remark: This problem illustrates that the concepts developed in this chapter can also be applied for a communication system that employs more than two waveforms.

5.26 (*Regenerative repeaters or why go digital*) In analog communication systems, amplifiers called repeaters are used to periodically boost the signal strength in transmission through a noise channel. However each amplifier also boosts the noise in the system. In contrast, digital communication systems allow us to detect and regenerate a clean (noise-free) signal in a transmission channel. Such devices, called *regenerative repeaters*, are frequently used in wireline and fiber optic communication channels. The front end of each regenerative repeater consists of a demodulator/detector that demodulates and detects the transmitted digital information sequence sent by the preceding repeater. Once detected, the sequence is passed to the transmitter side of the repeater, which maps the sequence into the signal waveforms that are transmitted to the next repeater.

Since a noise-free signal is regenerated at each repeater, the additive noise does not *accumulate*. However, when errors occur in the detector of a repeater, the errors are *propagated* forward to the following repeaters in the channel. To evaluate the effect of errors on the performance of the overall system, suppose that antipodal signalling is used, so that the probability of a bit error for one hop (signal transmission from one repeater to the next in the chain) is

$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{P5.38}$$

where E_b is the energy per bit, $N_0/2$ is the PSD of the noise and E_b/N_0 is the signal to noise ratio (SNR).

Since errors occur with low probability, we may ignore the probability that any one bit will be detected incorrectly more than once in transmission through a channel with K repeaters. Consequently, the number of errors will increase linearly with the number of regenerative repeaters in the channel, and therefore, the overall probability of error may be approximated as

$$P_b = KQ\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{P5.39}$$

In contrast, the use of K analog repeaters in the channel reduces the received SNR by K, and hence, the bit error probability is

$$P_b = Q\left(\sqrt{\frac{2E_b}{KN_0}}\right) \tag{P5.40}$$

Clearly, for the same probability of error performance, the use of the regenerative repeaters results in a significant saving in transmitted power compared with analog repeaters. Hence in digital communication systems, regenerative repeaters are preferable. Nevertheless, in wireline telephone channels that are used to transmit both analog and digital signals, analog repeaters are generally employed.

Now for the problem, consider a binary communication system that transmits data over a wireline channel of length 2000km. Repeaters are used every 40km to offset the effect of channel attenuation. The SPEC calls for a probability of bit error of 10^{-6}. Determine the SNR, E_b/N_0, required if:

(a) Analog repeaters are used.

(b) Regenerative repeaters are used.

5.27 (*Simulation of a binary communication system using antipodal signalling*) The probability of error, or bit-error-rate (BER), is an important performance parameter of any digital communication system. However, obtaining such a performance parameter in a closed-form expression is sometime very difficult, if not impossible. This is especially true for complex systems that employ error control coding, multiple access techniques, wireless transmission, etc. It is common in the study and design of digital communications that the BER is evaluated through computer simulation. In this assignment you shall use MATLAB to write a simple simulation program to test the BER performance of a binary communication system using antipodal signalling. The specific steps in your program are as follows:

- *Information Source*: Generate a *random* vector `b` that contains L information bits "0" and "1". The two bits should be equally likely. For this step, the functions `rand` and `round` in MATLAB might be useful.
- *Modulator*: The binary information bits contained in vector `b` are transmitted using antipodal signalling, where a voltage V is used for bit "1" and $-V$ for bit "0". Thus the transmitted signal is simply `y=V*(2*b-1);`
- *Channel*: The channel noise is additive white Gaussian noise with two-sided power spectral density $N_0/2 = 1$ (watts/Hz). The effect of this additive white Gaussian noise can be simulated in MATLAB by adding a noise vector `w` to the transmitted signal `y`. The vector of independent Gaussian noise samples with variance of 1 can be generated in MATLAB as follows: `w=randn(1,length(y));` The received signal `r` is simply `r=y+w;`

  Remark: In essence, the above implements a discrete (and equivalent) model of an antipodal signalling system. In particular, the simulated vector `r` is the output of the correlator or matched filter. Also for simplicity, it is assumed that $T_b = 1$ second.

- *Demodulator (or Receiver)*: With antipodal signalling, the demodulator is very simple. It simply compares the received signal with zero to make the decision.

Determine the minimum values of V to achieve the bit error rate (BER) levels of 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4} and 10^{-5}. Use each value of V you found to run your MATLAB program and record the actual BER. Plot (in log scale) both the theoretical and experimental BER versus V^2 (dB) on the same graph and compare.

Note that if you expect a BER of 10^{-K} for $K = 1, \ldots, 5$, then the length L of the information bit vector `b` should be at least $L = 100 \times 10^K$. This is because at least 100 erroneous bits need to be seen in each simulation run in order to have a reliable experimental BER value.

6

Baseband Data Transmission

6.1 Introduction

As pointed out in the previous chapter, binary digits (or bits) "0" and "1" are used simply to represent the information content. They are abstract (intangible) quantities and need to be converted into electrical waveforms for effective transmission or storage. How to perform such a conversion is generally governed by many factors, of which the most important one is the available transmission bandwidth of the communication channel or the storage media.

In baseband† data transmission, the bits are mapped into two voltage levels for direct transmission without any frequency translation. Such a baseband data transmission is applicable to cable systems (both metallic and fiber optics) since the transmission bandwidth of most cable systems is in the baseband. Various baseband signaling techniques, also known as line codes, have been developed to satisfy a number of criteria. The typical criteria are:

(i) *Signal interference and noise immunity:* Depending on the signal sets, certain signaling schemes exhibit superior performance in the presence of noise as reflected by the probability of bit error.

(ii) *Signal spectrum:* Typically one would like the transmitted signal to occupy as small a frequency band as possible. For baseband signaling, this implies a lack of high frequency components. However, it is sometimes also important to have no DC component. Having a signaling scheme which does not have a DC component implies that AC coupling via a transformer may be used in the transmission channel. This provides electrical isolation which

† Baseband modulation can be defined, somewhat imprecisely, as a modulation whose power spectrum density is huddled around $f = 0$ (Hz).

tends to reduce interference. Moreover, it is also possible in certain signaling schemes to match the transmitted signal to the special characteristics of a transmission channel.

(iii) *Signal synchronization capability:* In implementing the receiver, it is necessary to establish the beginning and the end of each bit transmission period. This typically requires a separate clock to synchronize the transmitter and the receiver. Self synchronization is however also possible if there are adequate transitions in the transmitted baseband signal. Several self-synchronizing baseband schemes have also been developed.

(iv) *Error detection capability:* Some signaling schemes have an inherent error detection capability. This is possible by introducing constraints on allowable transitions among the signal levels and exploiting those constraints at the receiver.

(v) *Cost and complexity of transmitter and receiver implementations:* This is still a factor which should not be ignored even though the price of digital logic continues to decrease.

This chapter discusses four baseband signaling schemes (also known as line codes) from the aspects of the first three criteria. The four signaling schemes are commonly known as nonreturn-to-zero-level (NRZ-L), return-to-zero (RZ), bi-phase-level or Manchester, and delay modulation or Miller.

6.2 Baseband Signaling Schemes

Nonreturn to Zero (NRZ) Code. The NRZ code can be regarded as the most basic baseband signaling scheme, since it appears "naturally" in synchronous digital circuits. In NRZ code, the signal alternates between the two voltage levels only when the current bit differs from the previous one.

Figure 6.1-(a) represents an example of NRZ waveform, where T_b is the bit duration. Note that there is only one polarity in the waveform, hence it is also known as unipolar NRZ waveform. This is the simplest version of NRZ and can be easily be generated. However the DC component of a long random sequence of ones and zeros is nonzero. More precisely, the DC component is $V\,\Pr[1_T] + 0\,\Pr[0_T] = VP_2 + 0P_1 = VP_2$. For the common case of equally likely bits, the DC component is one-half of the positive voltage, i.e., $0.5V$ (volts). Therefore it is common to pass the NRZ waveform through a level shifter. The resultant waveform

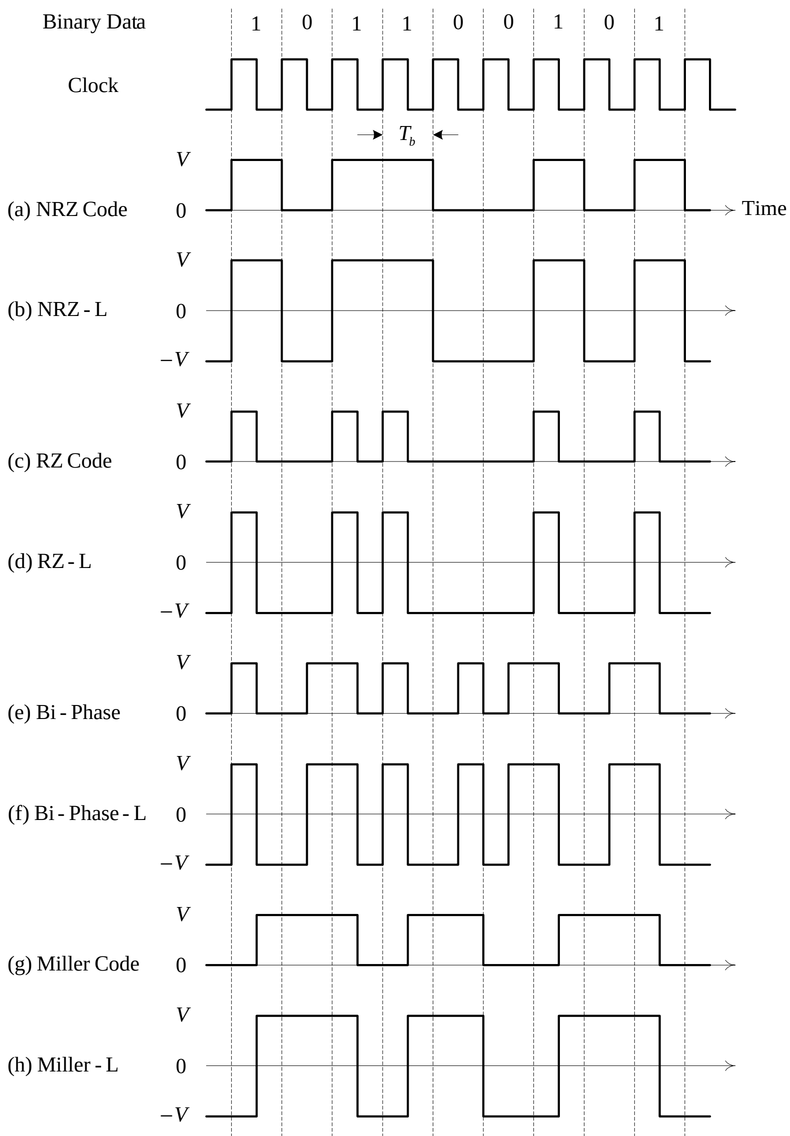


Fig. 6.1. Various binary signaling formats.

then alternates between $+V$ and $-V$ as shown in Figure 6.1-(b). It is called polar NRZ or NRZ-L waveform, whose DC component is $V\,\mathrm{Pr}[1]+$

$(-V)\,\Pr[0_T] = V(P_2 - P_1)$. Obviously, the DC component of the NRZ-L waveform is zero if the two bits are equally likely.

Observe that the NRZ-L code produces a transition whenever the current bit in the input sequence differs from the previous one. These transitions can be used for synchronization purposes at the receiver. However if the transmitted data contains long strings of similar bits, then the timing information is sparse, and regeneration of the clock signal at the receiver can be very difficult.

Return to Zero (RZ) Code. The RZ code is similar to the NRZ code except that the information is contained in the first half of the bit interval, while the second half is always at level "zero". An example of RZ waveform is shown in Figure 6.1-(c). Once again the code has a DC component, which is $(0.5V)\,\Pr[1_T] + 0\,\Pr[0_T] = (0.5V)P_2$. If $P_1 = P_2 = 0.5$, then the DC component is one-fourth of the positive voltage, i.e., $0.25V$ (volts). Figure 6.2 shows that the RZ code is generated by gating the basic NRZ signal with the transmitter clock.

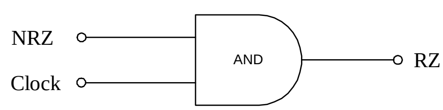


Fig. 6.2. RZ Encoder.

In order to fairly compare with the NRZ-L code in terms of error probability, we shall consider the RZ-L (or bipolar RZ) code where the two levels are $+V$ and $-V$ rather than V and 0. The corresponding waveform of RZ-L code is shown in Figure 6.1-(d). The DC component of RZ-L waveform is $-VP_1$, which is $-0.5V$ (volts) if $P_1 = P_2 = 0.5$. Thus the code still has a nonzero DC component.

Regarding timing information, note that as opposed to NRZ, with RZ a long string of bits "1" results in transitions from which a clock at the receiver can be regenerated. A string of bits "0", however, does not have any transitions just as the NRZ code. For this reason and the fact that it has poor spectral properties and inferior error performance, RZ coding is not used except in some very elementary transmitting and recording equipment.

Bi-phase (Biϕ) or Manchester Code. To overcome the poor synchronization capability of NRZ and RZ codes, bi-phase (Biϕ) coding has been developed. It encodes information in terms of level transitions in the middle of a bit interval. Note that this conversion of bits to electrical waveforms is in a sharp contrast to what done in NRZ and RZ codes, where the information bits are converted to voltage levels. The bi-phase conversion (or mapping, or encoding) rules are as follows:

- Bit "1" is encoded as a transition from a high level to a low level occurring in the middle of the bit interval.
- Bit "0" is encoded as a transition from a low level to a high level occurring in the middle of the bit interval.
- An additional "idle" transition may have to be added at the beginning of each bit interval to establish the proper starting level for the information carrying transition.

Examples of bi-phase and bi-phase-level waveforms are shown in Figures 6.1-(e) and (f), respectively. The DC component of the bi-phase signal is evaluated as $(0.5V)\,\Pr[1_T]+(0.5V)\,\Pr[0_T] = 0.5V[P_2+P_1] = 0.5V$, while the DC component of the bi-phase-level (Biϕ-L) signal is obviously 0. Note that the above results for the DC components hold regardless of the *a priori* probabilities of the bits. Therefore the bi-phase-level code does not have any DC component. Figure 6.3 shows that the bi-phase code can be generated with an XOR logic whose inputs are the basic NRZ signal and the transmitter clock.

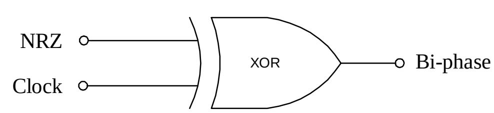


Fig. 6.3. Bi-phase encoder.

The bi-phase signal, however, occupies a wider frequency band than NRZ signal. This is due to the fact that for alternating bits there is one transition per bit interval while for two identical consecutive bits, two transitions occur per bit interval. On the other hand, because there is a predictable transition during every bit interval, the receiver can synchronize on that transition. The bi-phase code is thus known as a self-synchronizing code. It is commonly used in local area networks (LANs), such as the Ethernet.

Miller Code. This code is an alternative to the bi-phase code. It has at least one transition every two bit interval and there is never more than two transitions every two bit interval. It thus provides good synchronization capabilities, while requiring less bandwidth than the bi-phase signal. The encoding rules are:

- Bit "1" is encoded by a transition in the middle of the bit interval. Depending on the previous bit this transition may be either upward or downward.
- Bit "0" is encoded by a transition at the beginning of the bit interval if the previous bit is "0". If the previous bit is "1", then there is no transition.

The waveforms for Miller and Miller-level codes are illustrated in Figures 6.1-(g) and (h), respectively. The Miller-L signal can be generated from the NRZ signal by the circuit shown in Figure 6.4.

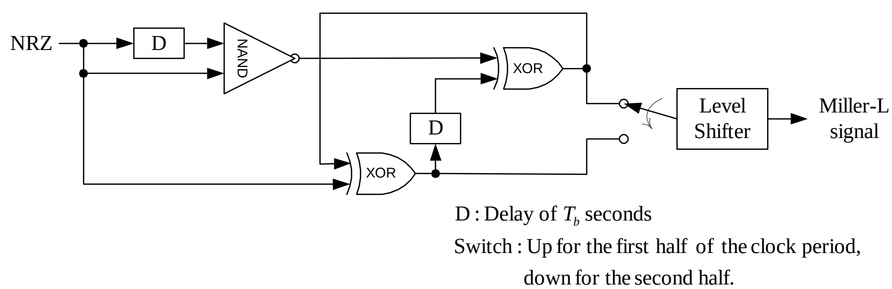


Fig. 6.4. Miller-L encoder.

6.3 Error Performance

To determine the probability of bit error for each of the line codes we shall consider that the transmitted signals are corrupted by zero-mean additive white Gaussian noise of spectral strength $\frac{N_0}{2}$ (watts/Hz) and that the two bits, "0" and "1", are *equally likely.* As shown in the previous chapter, the error probability of each line code is readily determined by identifying the elementary signals used for bits "0" and "1" and representing them in the signal space diagram. In all cases, the orthonormal basis set for the signal space can simply be determined by inspection. For each signaling scheme, a voltage swing from $-V$ to V volts is also assumed.

NRZ-L Code. The elementary signals are shown in Figure 6.5-(a), where each signal has energy $E_{\text{NRZ-L}} = V^2 T_b$ (joules). The single basis function and signal space plot are given in Figures 6.5-(b) and (c). Applying (5.105), the probability of bit error is given by,

$$\Pr[\text{error}]_{\text{NRZ-L}} = Q\left(\sqrt{2E_{\text{NRZ-L}}/N_0}\right) \tag{6.1}$$

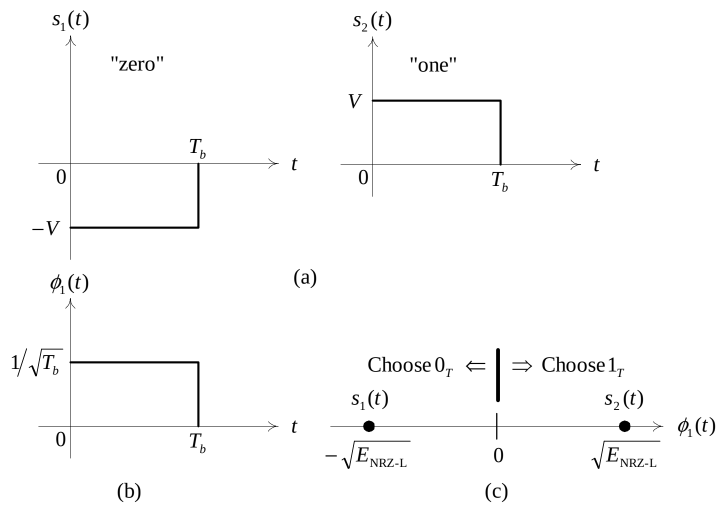


Fig. 6.5. NRZ-L code: (a) Elementary signals. (b) Basis function. (c) Signal space and decision regions.

RZ-L Code. Figure 6.6 shows the elementary signals, the basis functions, the signal space together with the optimum decision regions of RZ-L signaling. Each signal has energy $E_{\text{RZ-L}} = V^2 T_b = E_{\text{NRZ-L}}$ (joules) and the error probability is given as,

$$\Pr[\text{error}]_{\text{RZ-L}} = Q\left(\sqrt{E_{\text{RZ-L}}/N_0}\right) \tag{6.2}$$

Bi-phase-level (Biϕ-L) Code. Similar to the NRZ-L code, Biϕ-L code is also an antipodal signaling. Its elementary signals are, however, different in shape compared to that of NRZ-L code (so as to have self-synchronizing capability as discussed before). Here each signal has energy $E_{\text{Bi}\phi\text{-L}} = V^2 T_b = E_{\text{NRZ-L}}$ joules and the error probability is

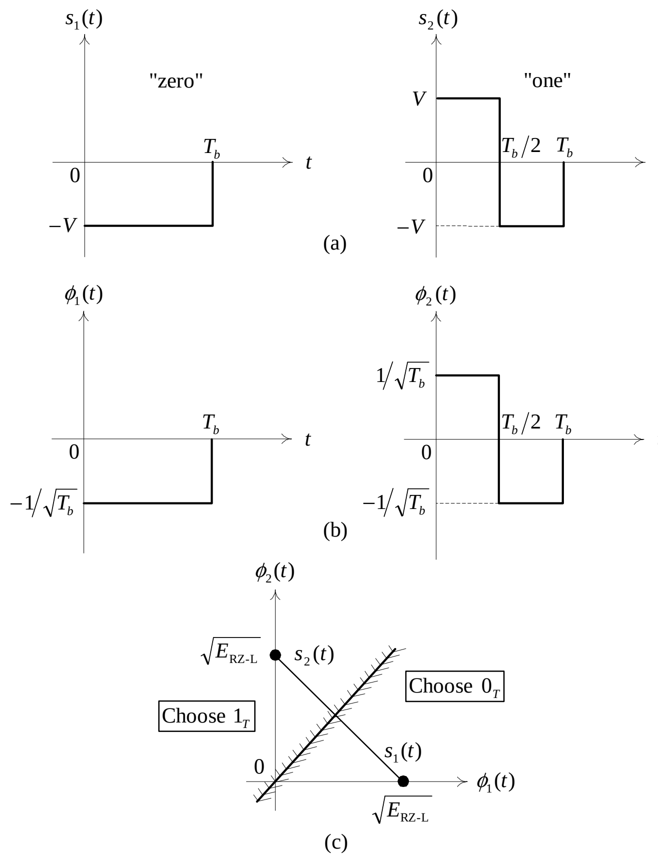


Fig. 6.6. RZ-L code: (a) Elementary signals. (b) Basis functions. (c) Signal space and decision regions.

expressed as

$$\Pr[\text{error}]_{\text{Bi}\phi\text{-L}} = Q\left(\sqrt{2E_{\text{Bi}\phi\text{-L}}/N_0}\right) \tag{6.3}$$

Miller-Level (M-L). For this code there are four elementary signals, two of which represent bits "1" with the other two representing bits "0". The four elementary signals are shown in Figure 6.8-(a), whereas the two orthonormal basis functions needed to represent these four signals are plotted in Figure 6.4-(b). The energy in each signal is $E_{\text{M-L}} = V^2 T_b = E_{\text{NRZ-L}}$ joules.

The minimum-distance receiver consists of projecting the received sig-

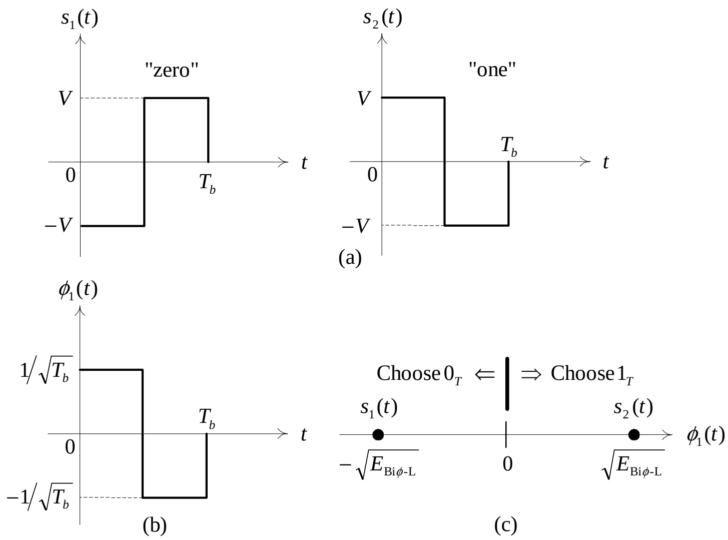


Fig. 6.7. Biϕ-L code: (a) Elementary signals. (b) Basis function. (c) Signal space and decision regions.

nal, $r(t)$, onto $\phi_1(t)$ and $\phi_2(t)$ which generates the statistics (r_1, r_2). Since each signal is equally likely and has a priori probability of $1/4$, the decision rule is to choose the signal to which the point (r_1, r_2) is closest. The decision space is shown in Figure 6.9.

To determine the bit error probability, note that

$$\Pr[\text{error}] = 1 - \Pr[\text{correct}] \tag{6.4}$$

where

$$\Pr[\text{correct}] = \sum_{i=1}^{4} \Pr[s_i(t)] \Pr[\text{correct}|s_i(t)] \tag{6.5}$$

Also $\Pr[s_i(t)] = 1/4$, $i = 1, 2, 3, 4$. Because of the symmetry, $\Pr[\text{correct}|s_i(t)]$ is the same regardless of which signal is considered. Therefore

$$\Pr[\text{correct}] = \Pr[\text{correct}|s_i(t)] \tag{6.6}$$

The above probability can be found by evaluating the volume under the probability density function $f(r_1, r_2|s_1(t))$. Consider the volume under $f(r_1, r_2|s_1(t))$ over region 1 shown as the shaded area in Figure 6.10. The random variables r_1 and r_2 are statistically independent Gaussian

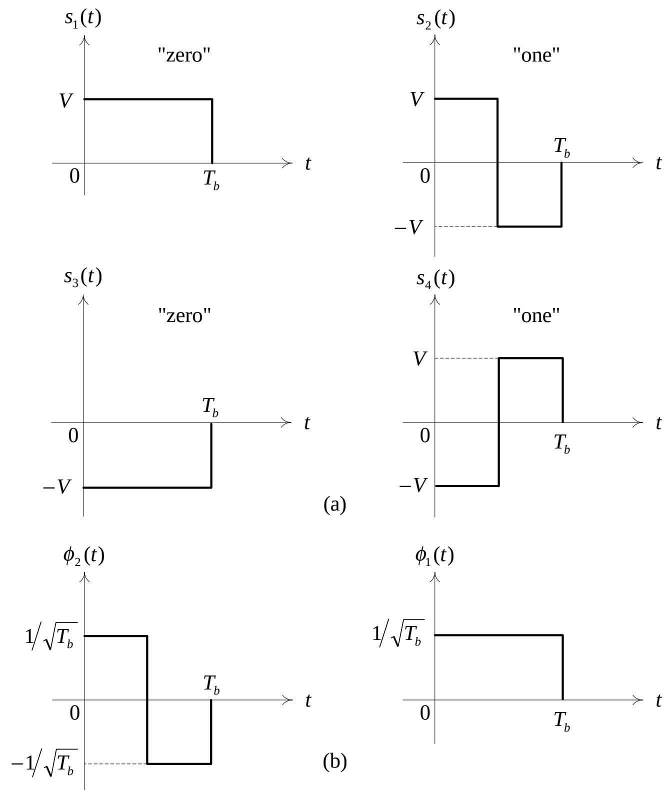


Fig. 6.8. Miller-L code: (a) Elementary signals. (b) Basis functions.

random variables, with means of 0 and $\sqrt{E_{\text{M-L}}}$ (volts), respectively, and with the same variance of $\frac{N_0}{2}$ (watts).

Rather than evaluating the integral with r_1, r_2 as variables we change variables so that the integral is expressed in terms of $\hat{r}_1, \hat{r}_2$ which are also statistically independent Gaussian random variables with the same mean $\sqrt{E_{\text{M-L}}/2}$ and variance $\frac{N_0}{2}$. This is much more convenient since the region of the double integrals can be expressed as two independent

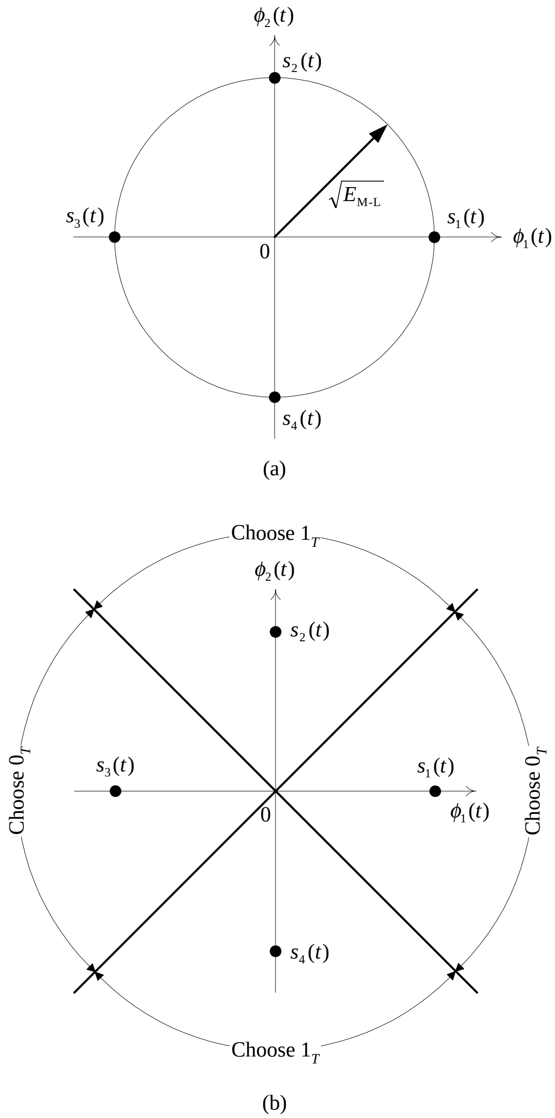


Fig. 6.9. Miller-L code: (a) Signal space plot. (b) Decision regions.

regions of variables $\hat{r}_1, \hat{r}_2$. In particular,

$$
\begin{aligned}
\Pr[\text{correct}|s_i(t)] &= \int_0^\infty \int_0^\infty f\left(\hat{r}_1, \hat{r}_2 | s_i(t)\right) \mathrm{d}\hat{r}_1 \mathrm{d}\hat{r}_2 \\
&= \left[\int_0^\infty \frac{1}{\sqrt{\pi N_0}} \exp\left\{ \frac{\left(\hat{r} - \sqrt{E_{\text{M-L}}/2}\right)^2}{N_0} \right\} \mathrm{d}\hat{r} \right]^2 \\
&= \left[1 - Q\left(\sqrt{E_{\text{M-L}}/N_0} \right) \right]^2 \qquad (6.7)
\end{aligned}
$$

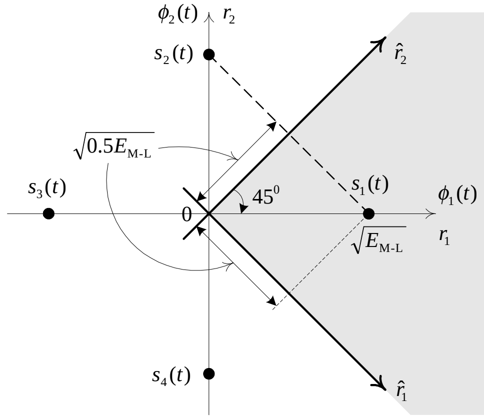


Fig. 6.10. Evaluating the error probability for Miller-L code.

The bit error probability for the Miller code is therefore given by

$$\begin{aligned} \Pr[\text{error}]_{\text{M-L}} &= 1 - \left[1 - Q\left(\sqrt{E_{\text{M-L}}/N_0}\right)\right]^2 \\ &= 2Q\left(\sqrt{E_{\text{M-L}}/N_0}\right) - \left[Q\left(\sqrt{E_{\text{M-L}}/N_0}\right)\right]^2 \quad (6.8) \end{aligned}$$

Because the typical value for the error probability is 10^{-6} or less, the second term in (6.8) is insignificant compared to the first term and (6.8) can be well approximated by

$$\Pr[\text{error}]_{\text{M-L}} \approx Q\left(\sqrt{E_{\text{M-L}}/N_0}\right) \quad (6.9)$$

where the factor 2 is also ignored.

Comparing the four signaling schemes investigated in this chapter we see that they have the same energy per bit, i.e.,

$$E_{\text{NRZ-L}} = E_{\text{RZ-L}} = E_{\text{Bi}\phi\text{-L}} = E_{\text{M-L}} = V^2T_b \equiv E_b \text{ (joules/bit)}. \quad (6.10)$$

Therefore, in terms of energy per bit, E_b, the error probabilities for the four signaling schemes are:

$$\Pr[\text{error}]_{\text{NRZ-L}} = \Pr[\text{error}]_{\text{Bi}\phi\text{-L}} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \quad (6.11)$$

$$\Pr[\text{error}]_{\text{RZ-L}} = \Pr[\text{error}]_{\text{M-L}} = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \quad (6.12)$$

These expressions are plotted in Figure 6.11 as functions of E_b/N_0. What the above expressions say is that the signal-to-noise ratio (SNR), E_b/N_0, would have to be double (i.e., it requires 3 dB more transmitted power) for Miller-L or RZ-L coding to achieve the same error probability as NRZ-L or Biϕ-L coding. This fact can also be verified from Figure 6.11 at high SNR region.

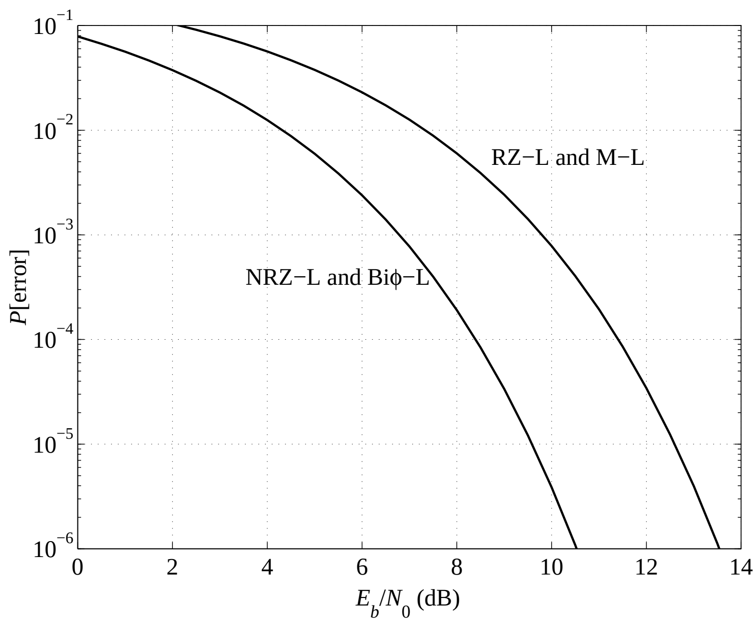


Fig. 6.11. Probabilities of error for various baseband signaling schemes.

6.4 Optimum Sequence Demodulation for Miller Signaling

To this point the demodulation developed in Chapter 5 and applied in this chapter to the four line codes is a *symbol-by-symbol* demodulator. This type of demodulator assumes that there is no information about the transmitted sequence other than that in the present bit interval. Though this is true for the first three line codes it does not hold for Miller signaling. Miller modulation has memory, since the transmitted signal in the present bit interval depends on both the present bit and also the previous bit. In demodulation of the received signal knowledge of this memory can, and should be, exploited to make a decision. Rather than symbol-by-symbol demodulation, this leads to sequence demodulation.

Before discussing sequence demodulation, a simple example is given to clearly illustrate the shortcoming of the symbol-by-symbol receiver.

Example 6.1. Assume that the four signals $s_i(t)$, $i = 1, 2, 3, 4$ have unit energy. The projections of the received signals on to $\phi_1(t)$ and $\phi_2(t)$, denoted by $r_1^{(k)}, r_2^{(k)}$ where k signifies the bit interval, are given as $r_1^{(1)} = -0.2$, $r_2^{(1)} = -0.4$, $r_1^{(2)} = +0.2$, $r_2^{(2)} = -0.8$, $r_1^{(3)} = -0.61$, $r_2^{(3)} = +0.5$, $r_1^{(4)} = -1.1$, $r_2^{(4)} = +0.1$. The distances squared from the received signal to all four possible transmitted signal in each bit interval are tabulated in Table 6.1. Thus the symbol-by-symbol minimum-distance receiver decides $\{s_4(t), s_4(t), s_3(t), s_3(t)\}$ as the sequence of transmitted signals, which corresponds to the bit sequence $\{1100\}$. However, according to the encoding rule of Miller modulation, the sequence $\{s_4(t), s_4(t), s_3(t), s_3(t)\}$ is not a *valid* transmitted sequence. This implies that there must be error in the above symbol-by-symbol decision rule.

Table 6.1. *Distances squared from the received signals to four possible transmitted signals.*

Transmitted signal	Distance squared			
	$0 \to T_b$	$T_b \to 2T_b$	$2T_b \to 3T_b$	$3T_b \to 4T_b$
$s_1(t)$	1.6	1.28	2.8421	4.42
$s_2(t)$	2.0	3.28	0.6221	2.02
$s_3(t)$	0.8	2.08	**0.4021**	**0.02**
$s_4(t)$	**0.4**	**0.08**	2.6221	2.42

The above example suggests that a better decision rule could be achieved by exploiting the memory of the Miller code. One possible way to demodulate the received signal to an allowable transmitted sequence is described next. Assume that a total of n bits are transmitted. Each n-bit pattern results in a transmitted signal over $0 \le t \le nT_b$. Obviously, the total number of different bit patterns (or signals) is $M = 2^n$, i.e., it grows exponentially with n. Typically n is a large number and

2^n can be a very very big number.† Denote the entire transmitted signal over the time interval $[0, nT_b]$ as $S_i(t)$, $i = 1, 2, \ldots, M = 2^n$. The signal $S_i(t)$ can also be written as $S_i(t) = \sum_{j=1}^{n} S_{ij}(t)$ where $S_{ij}(t)$ is one of the four possible signals used in Miller code in the bit interval $[(j-1)T_b, jT_b]$ and zero elsewhere. Note the meaning of the subscript notation, i refers to the specific transmitted signal under consideration and j to the Miller signal in the jth bit interval. At the receiver, the received signal over the time interval $[0, nT_b]$ is $r(t)$ and it can also be written as $r(t) = \sum_{j=1}^{n} r_j(t)$ where $r_j(t) = r(t)$ in the interval $[(j-1)T_b, jT_b]$ and zero elsewhere.

To decide which of the M possible signals that has transmitted one can compute the distance from each signal $S_i(t)$ to the received signal $r(t)$, which is simply

$$d_i = \sqrt{\int_0^{nT_b} [r(t) - S_i(t)]^2 \mathrm{d}t}. \tag{6.13}$$

After such distances are computed for all $S_i(t)$, $i = 1, 2, \ldots, M = 2^n$, the decision rule is to choose as the transmitted signal the one that is closest to $r(t)$.

The computation of the distance in the above decision rule can be simplified by projecting the continuous-time transmitted and received signals onto the signal space of Miller code. To this end, proceed as follows. First, note that if the distance is minimum then the square of it will also be minimum. Then compute d_i^2 by splitting the integral up as follows

$$d_i^2 = \sum_{j=1}^{n} \int_{(j-1)T_b}^{jT_b} [r_j(t) - S_{ij}(t)]^2 \mathrm{d}t. \tag{6.14}$$

Now the term of summation $d_{ij}^2 = \int_{(j-1)T_b}^{jT_b} [r_j(t) - S_{ij}(t)]^2 \mathrm{d}t$ is simply the distance squared of $r_j(t)$ to $S_{ij}(t)$. Let $[r_1^{(j)}, r_2^{(j)}]$ be the outputs of the two correlators (or matched filters) in the $[(j-1)T_b, jT_b]$ time interval and $[S_{i1}^{(j)}, S_{i2}^{(j)}]$ the coefficients in the representation of $S_{ij}(t)$. We know that $d_{ij}^2 = \left(r_1^{(j)} - S_{i1}^{(j)}\right)^2 + \left(r_2^{(j)} - S_{i2}^{(j)}\right)^2$. Therefore, the distance computation in (6.13) can be rewritten as:

$$d_i^2 = \sum_{j=1}^{n} \left[\left(r_1^{(j)} - S_{i1}^{(j)}\right)^2 + \left(r_2^{(j)} - S_{i2}^{(j)}\right)^2 \right] \tag{6.15}$$

† Unless you are a loan shark or a banker (which may amount to the same thing) exponential growth is *bad*, while linear growth is *good*.

If the transmitted bits are equally likely and if the channel is AWGN, it can be shown that (see Chapter 7) the above decision rule, which is based on the minimum sequence distance, constitutes an optimum receiver for Miller signaling. It is optimum in the sense that the probability of making a sequence error is minimized.

Viterbi Algorithm. Though the optimum decision rule is rather simple in its interpretation, it requires an extensive and simply impossible amount of computations if direct evaluations of $M = 2^n$ distances as in (6.15) are to be carried out. Fortunately, a much more efficient algorithm, due to A. J. Viterbi and known as the Viterbi algorithm, exists to implement the optimum decision rule. At the heart of the Viterbi algorithm are the concepts of the state and trellis diagrams that are used to elegantly represent all the possible transmitted sequences.

In general, the *state* of a system can be looked upon very simply (and perhaps somewhat loosely) as what information from the past do we need at the present time, which together with the present input allows us to determine the system's output for any future input. Here consider the system to be the modulator that produces the Miller encoded signal. Recall that in Miller signaling, the transmitted signal depends on the bit to be transmitted in the present interval and the signal, or bit, transmitted in the previous interval. There are four possible combinations of the present bit and previous signal and therefore there are four states. The *state diagram* that represents the Miller encoding rule is shown in Figure 6.12.

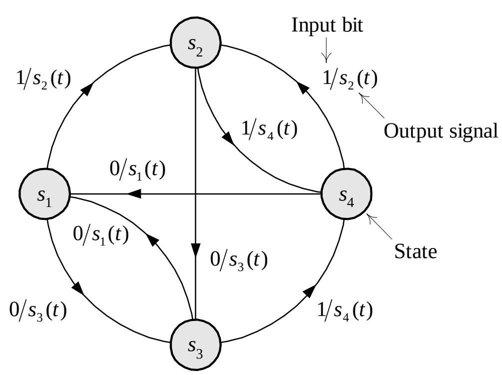


Fig. 6.12. State diagram of the Miller code. The state is defined as the signal transmitted in the previous bit interval.

Although the state diagram in Fig. 6.12 clearly and concisely describes the encoding rule, it only describes the rule in a single bit interval. To illustrate the modulator's output for any possible input sequence one can follow the path dictated by the input bits and produce the output signal. However, a more informative approach is to use a *trellis diagram*. In essence, a trellis diagram is simply an unfolded state diagram. Figure 6.13 shows the trellis diagram for Miller modulation over the interval $[0, 4T_b]$.

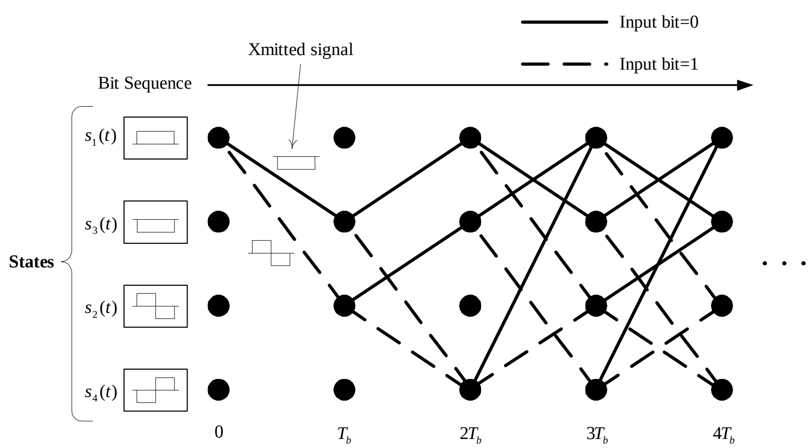


Fig. 6.13. Trellis diagram of Miller code.

Some important remarks regarding Figure 6.13 are as follows:

- It is assumed that the initial (or starting) state is $s_1(t)$. In practice, this can always be guaranteed by an agreed protocol.
- A transmitted signal in a given bit interval is represented by a *branch* connecting two states. The *solid* line corresponds to bit "0", whereas the *dashed* line corresponds to bit "1".
- In the fourth bit interval ($[3T_b, 4T_b]$) the trellis is fully expanded. From this bit interval on, the trellis pattern is the same.
- Each possible output sequence is represented by a *path* through the trellis. Conversely, each path in the trellis represents a valid (or allowable) Miller signal.

Working with the trellis diagram, the main steps in the Viterbi algorithm to find the sequence (i.e., the path through the trellis) that is closest to the received signal are as follows

Step 1: Start from the initial state ($s_1(t)$ in our case).

Step 2: In each bit interval, calculate the *branch metric*, which is the distance squared between the received signal in that interval with the signal corresponding to each possible branch. Add this branch metric to the previous metrics to get the *partial path metric* for each partial path up to that bit interval.

Step 3: If there are two partial paths entering the same state, discard the one that has a larger partial path metric and call the remaining path the *survivor*.

Step 4: Extend only the survivor paths to the next interval. Repeat Steps 2 to 4 till the end of the sequence.

In Step 3 above, the procedure where one of the competing partial paths is discarded at each state requires some justification. Namely, nothing that is received in the future will give one any information about what happened in the past. This is because future noise samples are statistically independent of present ones (recall that the noise is white and Gaussian) and also the bits are assumed to be statistically independent.

Example 6.2. To illustrate the Viterbi algorithm, let's revisit Example 6.1 and apply the Viterbi algorithm to demodulate the received sequence.

For the first interval there are only two possibilities. The distance squared of the received signal to these two possibilities are computed but no decision is made. This is shown in Figure 6.14.

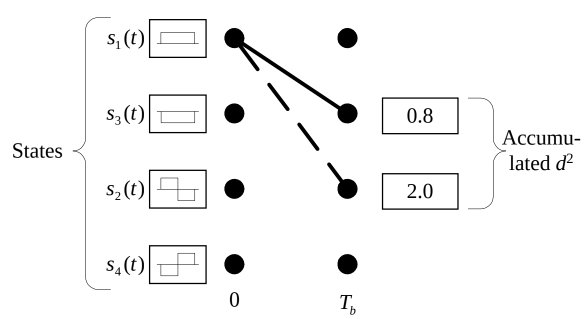


Fig. 6.14. Accumulated distance squared in the first bit interval.

At this point no decision is made. Rather we continue to the next interval and compute the distance squared of the received signal (in the interval $[T_b, 2T_b]$) to the signal along a particular branch. The squared distance of each possible transmitted sequence to the received sequence is:

$$00 \rightarrow 2.08;\ 10 \rightarrow 4.08;\ 01 \rightarrow 0.88;\ 11 \rightarrow 2.08.$$

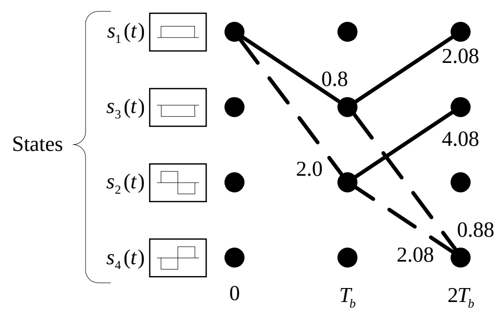


Fig. 6.15. Accumulated squared distances after the second bit interval

If we were forced (or inclined) to make a decision at this stage we would choose sequence 01 for a rather "obvious" reason. Note that symbol-by-symbol demodulation which ignores the memory results in the sequence 11 being chosen. Again we do not make any firm decisions but proceed to the next interval. We do, however, make one decision. For the two sequences that end in state $s_4(t)$ we discard one, namely, sequence 11. The picture now looks as in Figure 6.16.

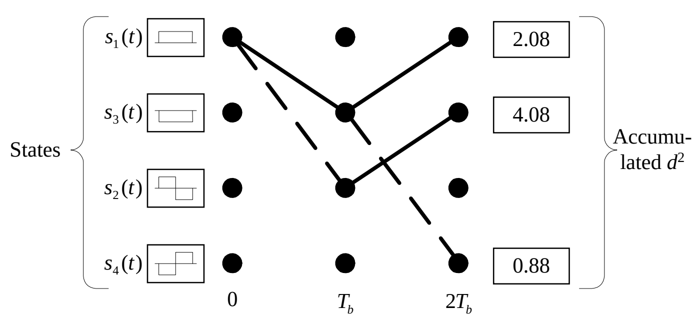


Fig. 6.16. Squared distances of the survivor paths after the second bit interval.

In the third bit interval, the squared distance of the received signal to each possible signal during this bit interval (called the *branch metric*) is now computed (see Table 6.1), added to the surviving squared distances and new surviving sequences are determined. After the third bit interval, the surviving sequences are:

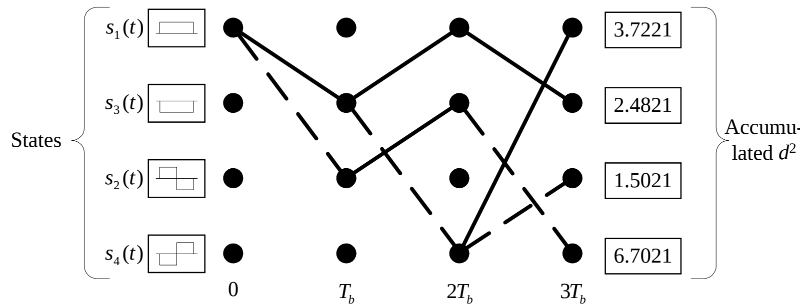


Fig. 6.17. Survivor paths and accumulated squared distances after the third bit interval.

For the fourth bit interval, the branch squared distances are calculated as in Table 6.1 (see column 5). Add these distances appropriately to the survivors after the third bit interval. For the two sequences that converge in a given state, choosing the sequence that is closest to the received signal results in the following survivors.

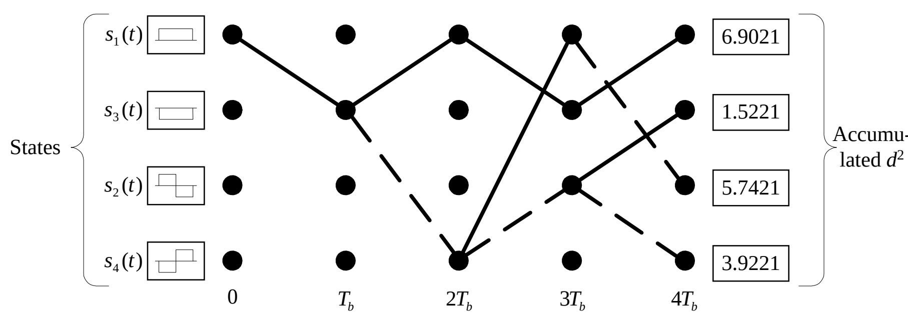


Fig. 6.18. Accumulated distance squared after the fourth bit interval.

So, the surviving sequences are (from top to bottom):

$$
\begin{aligned}
&s_3(t), s_1(t), s_3(t), s_1(t) \quad \Leftrightarrow \quad 0000\\
&s_3(t), s_4(t), s_2(t), s_3(t) \quad \Leftrightarrow \quad 0110\\
&s_3(t), s_4(t), s_1(t), s_2(t) \quad \Leftrightarrow \quad 0101\\
&s_3(t), s_4(t), s_2(t), s_4(t) \quad \Leftrightarrow \quad 0111
\end{aligned}
$$

We could continue ad nauseum, i.e., until we are ready to make a decision on the transmitted sequence or we know (by prior agreement or by protocol) that the sequence transmission is ended. Note that at this time ($4T_b$) one can make a very *firm decision* about the transmitted sequence, i.e., a decision that will not change due to future received signals. This decision is that of the first transmitted bit being "0", which corresponds to the common stem over $[0, T_b]$ of all the four survivor paths.

From the above surviving sequences, it is quite obvious that if one needs to make a decision at $4T_b$, one shall decide that the sequence $s_3(t), s_4(t), s_2(t), s_3(t)$, which corresponds to 0110, was transmitted.

6.5 Spectrum

The transmitted power required by a modulation scheme to achieve a certain error performance is important but equally important is the bandwidth requirement of the modulation. Therefore it is necessary to obtain the power spectrum density. Here we look at NRZ-L, bi-phase and Miller coding. Only the spectrum for NRZ-L signal is derived from basic principles. A more general approach to spectrum derivation for statistically independent binary modulation is given in the next chapter. Since their derivations are fairly involved, the bi-phase and Miller spectra are simply given.

For each signaling scheme, let the energy in an elementary waveform be $E \equiv V^2 T_b$ joules. Also let the a priori probability of bit "1" be $P_2 = P$ and of bit "0" be $P_1 = 1 - P$. As usual, we assume that the bits in different intervals are statistically independent.

NRZ-L Code. A typical NRZ-L signal is shown in Figure 6.19, where the starting time t_0 is a random variable, uniformly distributed in the time interval $[0, T_b]$. The pdf of the random variable $\mathbf{t}_0$ is plotted in Figure 6.20.

To determine the power spectrum density of $s(t)$ we first determine the autocorrelation function $R_{\mathbf{s}}(\tau)$ of $\mathbf{s}(t)$ which is the average value of the product $\mathcal{E}\{\mathbf{s}(t)\mathbf{s}(t+\tau)\}$, over the ensemble of all possible time signals. The power spectrum density is the Fourier transform of the autocorrelation function.

To start, consider the following two special cases for the delay τ:

1) $\tau = 0$. With this value of τ, $\mathcal{E}\{\mathbf{s}(t)\mathbf{s}(t+\tau)\}$ gives the mean-square

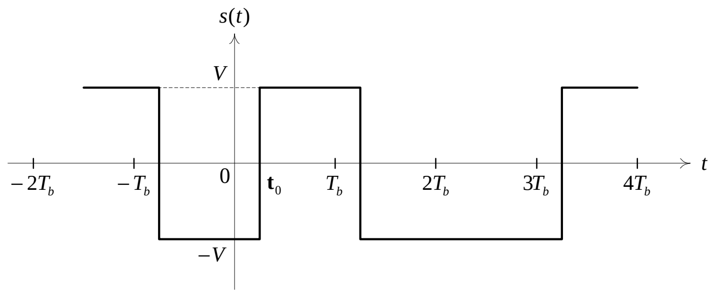


Fig. 6.19. A typical waveform of NRZ-L signaling.

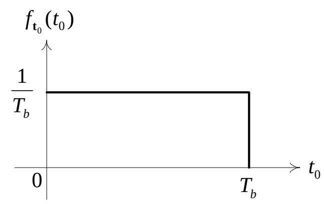


Fig. 6.20. Probability density function of the starting time $\mathbf{t}_0$.

value of the signal, i.e.,

$$\begin{aligned}\mathcal{E}\{\mathbf{s}^2(t)\} &= V^2\,\Pr[1_T] + (-V)^2\,\Pr[0_T] \\ &= V^2(P_1 + P_2) \\ &= V^2 \text{ (watts)} \end{aligned} \tag{6.16}$$

which is expected.

2) $\tau > T_b$. Note that for a fixed t, the random variables $\mathbf{s}(t)$ and $\mathbf{s}(t+\tau)$ are statistically independent random variables which take on the value of V or $-V$ with a probability of P_2 or P_1 respectively. The average value of the product is thus given by

$$\begin{aligned}\mathcal{E}\{\mathbf{s}(t)\mathbf{s}(t+\tau)\} &= (V)(V)\,\Pr[1_T \text{ at } t \text{ and } 1_T \text{ at} (t+\tau)] \\ &\quad +(-V)(-V)\,\Pr[0_T \text{ at } t \text{ and } 0_T \text{ at } (t+\tau)] \\ &\quad +(V)(-V)\,\Pr[1_T \text{ at } t \text{ and } 0_T \text{ at } (t+\tau)] \\ &\quad +(-V)(V)\,\Pr[0_T \text{ at } t \text{ and } 1_T \text{ at } (t+\tau)] \\ &= V^2P_2^2 + V^2P_1^2 - V^2P_2P_1 - V^2P_1P_2 \end{aligned} \tag{6.17}$$

where we have used the fact that a bit transmitted at time t is statistically independent of that at time $t+\tau$, for $\tau > T_b$. Let $P_2 = P$ and

$P_1 = 1 - P$, (6.17) reduces to

$$\begin{aligned} \mathcal{E}\{\mathbf{s}(t)\mathbf{s}(t+\tau)\} &= V^2[P^2 + (1-P)^2 - 2P(1-P)] \\ &= V^2(1-2P)^2, \quad \tau > T_b \end{aligned} \tag{6.18}$$

Thus far we know the autocorrelation function for $t = 0$ and $\tau > T_b$. We still need to determine $R_{\mathbf{s}}(t)$ over the interval $0 \le \tau \le T_b$ (recall that $R_{\mathbf{s}}(-\tau) = R_{\mathbf{s}}(\tau)$). To this end, consider the average value of the product $\mathbf{s}(t)\mathbf{s}(t+\tau)$ at two time instants t and $t+\tau$, where $\tau < T_b$. Consider the two possibilities for $\mathbf{t}_0$ (see Figure 6.21):

(a) Either $\mathbf{t}_0 < 0$ or $\mathbf{t}_0 > t+\tau$. For this case, the product $\mathbf{s}(t)\mathbf{s}(t+\tau) = V^2$ regardless of whether bit “1” or bit “0” is transmitted. This is because t and $t+\tau$ fall within the same bit interval.

(b) $t < \mathbf{t}_0 < t+\tau$, i.e., t falls within one bit interval while $t+\tau$ falls in another interval. In this case, the following are the different possibilities for $\mathbf{s}(t)$ and $\mathbf{s}(t+\tau)$:

$$\begin{aligned} \mathbf{s}(t) &= V; & \mathbf{s}(t+\tau) &= V \\ \mathbf{s}(t) &= V; & \mathbf{s}(t+\tau) &= -V \\ \mathbf{s}(t) &= -V; & \mathbf{s}(t+\tau) &= V \\ \mathbf{s}(t) &= -V; & \mathbf{s}(t+\tau) &= -V \end{aligned} \tag{6.19}$$

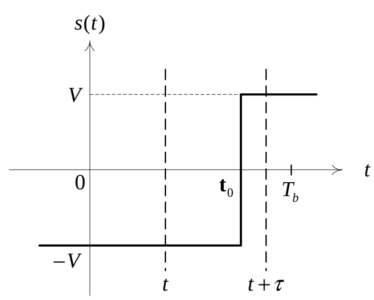


Fig. 6.21. Two time instants t and $t+\tau$ under consideration.

Now, the average value of the product, namely $\mathcal{E}\{\mathbf{s}(t)\mathbf{s}(t+\tau)\}$, can be determined as follows:

Case (a):

$$\mathcal{E}\{\mathbf{s}(t)\mathbf{s}(t+\tau)\} = V^2 \Pr[\mathbf{t}_0 < t \text{ or } \mathbf{t}_0 > t+\tau] \Pr[1_T] + V^2 \Pr[\mathbf{t}_0 < t \text{ or } \mathbf{t}_0 > t+\tau] \Pr[0_T] \tag{6.20}$$

Case (b):

$$\begin{aligned}\mathcal{E}\{\mathbf{s}(t)\mathbf{s}(t+\tau)\} &= V^2\,\Pr[t<\mathbf{t}_0<t+\tau]\,\Pr[1_T \text{ at } t]\,\Pr[1_T \text{ at } (t+\tau)]\\ &\quad +V^2\,\Pr[t<\mathbf{t}_0<t+\tau]\,\Pr[0_T \text{ at } t]\,\Pr[0_T \text{ a t}(t+\tau)]\\ &\quad -V^2\,\Pr[t<\mathbf{t}_0<t+\tau]\,\Pr[1_T \text{ at } t]\,\Pr[0_T \text{ at } (t+\tau)]\\ &\quad -V^2\,\Pr[t<\mathbf{t}_0<t+\tau]\,\Pr[0_T \text{ at } t]\,\Pr[1_T \text{ at } (t+\tau)]\end{aligned} \tag{6.21}$$

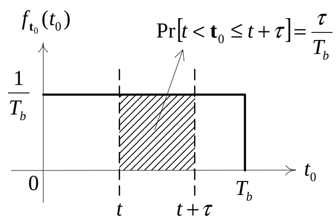


Fig. 6.22. Probability that $t<\mathbf{t}_0<t+\tau$.

The probability that $t<\mathbf{t}_0<t+\tau$ is given by the shaded area in Figure 6.22, while $\Pr[\mathbf{t}_0<t$ or $\mathbf{t}_0>t+\tau]=1-\Pr[t<\mathbf{t}_0<t+\tau]=1-\tau/T_b$. Therefore

$$\begin{aligned}R_{\mathbf{s}}(\tau) &= \mathcal{E}\{\mathbf{s}(t)\mathbf{s}(t+\tau)\}\\ &= V^2P_2(1-\tau/T_b)+V^2P_1(1-\tau/T_b)\\ &\quad +V^2P_2^2(\tau/T_b)+V^2P_1^2(\tau/T_b)\\ &\quad -V^2P_2P_1(\tau/T_b)-V^2P_1P_2(\tau/T_b), \quad 0<t<T_b\end{aligned} \tag{6.22}$$

which reduces to $R_{\mathbf{s}}(\tau)=V^2\left[1-\frac{4\tau}{T_b}P(1-P)\right]$, $0<t<T_b$. The complete expression for $R_{\mathbf{s}}(\tau)$ is as follows:

$$R_{\mathbf{s}}(\tau)=\begin{cases} V^2\left[1-\dfrac{4|\tau|}{T_b}P(1-P)\right], & |\tau|<T_b\\ V^2(1-2P)^2, & |\tau|>T_b\end{cases} \tag{6.23}$$

which plots as in Figure 6.23.

Finally, the power spectrum density is found as follows:

$$\mathcal{P}_{\mathbf{s}}(t)=\mathcal{F}\{R_{\mathbf{s}}(\tau)\}=V^2\left\{(1-2P)^2\delta(f)+4P(1-P)T_b\frac{\sin^2(\pi fT_b)}{(\pi fT_b)^2}\right\} \tag{6.24}$$

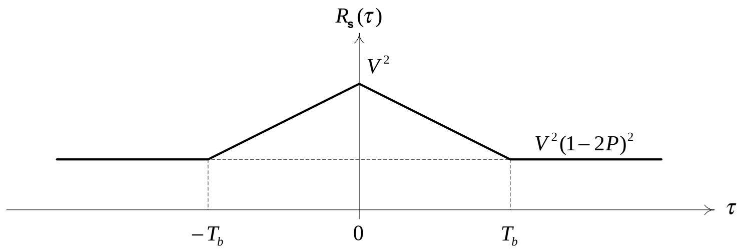


Fig. 6.23. Autocorrelation function $R_{\mathbf{s}}(\tau)$.

Normalizing with respect to the energy $E = V^2 T_b$ yields

$$\frac{\mathcal{P}_s(f)}{E} = \frac{1}{T_b}(1-2P)^2\delta(f) + 4P(1-P)\frac{\sin^2(\pi f T_b)}{(\pi f T_b)^2} \tag{6.25}$$

Note that when $P = 0.5$ the impulse at $f = 0$ disappears as expected.

Bi-phase Code. The spectrum is given by

$$\frac{\mathcal{P}_s(f)}{E} = \frac{1}{T_b}(1-2P)^2 \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{2}{n\pi}\right)^2 \delta\left(f - \frac{n}{T_b}\right) + 4P(1-P)\frac{\sin^4(\pi f T_b/2)}{(\pi f T_b/2)^2} \tag{6.26}$$

There is no DC component as expected. Further if $P = 0$ or $P = 1$, the continuous part of the spectrum disappears and there is only a line spectrum, reflecting the fact that the signal is periodic. When $P = 0.5$ the line spectrum disappears and only the continuous spectrum is present.

Miller Code. The expression for the spectrum of Miller code is

$$\frac{\mathcal{P}_s(f)}{E} = \frac{1}{2\theta^2(17+8\cos 8\theta)}(23 - 2\cos\theta - 22\cos 2\theta - 12\cos 3\theta + 5\cos 4\theta + 12\cos 5\theta + 2\cos 6\theta - 8\cos 7\theta + 2\cos 8\theta) \tag{6.27}$$

where $\theta = \pi f T_b$ and $P_2 = P_1 = 0.5$.

For comparison, the plots of the three different power spectral densities are shown in Figure 6.24. Observe that the Miller code has several advantages in terms of spectral properties over that of NRZ-L or bi-phase code. These are:

- The majority of the signaling energy lies at frequencies less than one-half the bit rate $f_b = \frac{1}{T_b}$.

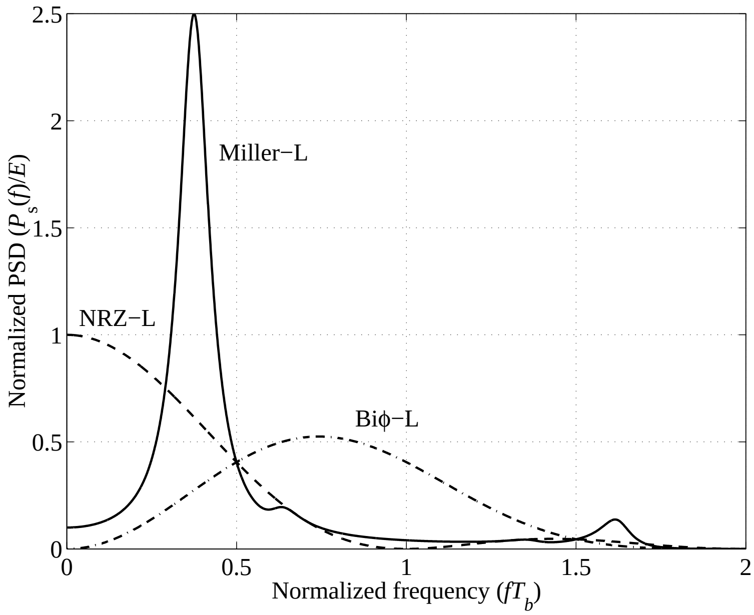


Fig. 6.24. Power spectral densities of NRZ-L, Biϕ-L and Miller-L signals.

- The spectrum is small in the vicinity of $f = 0$. This is important for channels which have poor DC response (for example, in magnetic recording).
- The Miller code is insensitive to the 180^0 phase ambiguity common to NRZ-L and bi-phase coding.
- Bandwidth requirement is approximately 1/2 of that needed by bi-phase coding.

6.6 Summary

The four baseband modulation methods dealt with in this chapter though basic are still found, and indeed should continue to be found in digital communication systems. This holds, in particular, for NRZ-L, Biphase and Miller modulation. The next chapter considers passband modulation techniques. Though the approach taken there is a direct one, passband modulation can also be considered to be the frequency translation of an equivalent baseband modulation. This is also touched upon in the next chapter.

Miller modulation has memory and this property led to the important concepts of state, trellis, sequence demodulation and Viterbi's algorithm. The next three chapters continue the study of memoryless modulation techniques since they are important not only in their own right, but are found in modulation paradigms that have memory. State, trellis and sequence modulation/demodulation are returned to the chapter on bandlimited channels where intersymbol interference (ISI) is present and the one on trellis-coded modulation (TCM).

Problems

6.1 (*Miller coding*) Consider the optimum decision rule of Miller code that is based on the minimum sequence distance.

(a) Show that finding the sequence with the minimum distance as computed in (6.13) is equivalent to finding the sequence with the maximum correlation metric, computed as follows:

$$\int_0^{nT_b} r(t)S_i(t)\mathrm{d}t = \sum_{j=1}^{n}[r_1^{(j)}, r_2^{(j)}]\begin{bmatrix} S_{i2}^{(j)} \\ S_{i1}^{(j)} \end{bmatrix} \tag{P6.1}$$

(b) Clearly describe all the steps of the Viterbi algorithm to find the transmitted sequence based on the maximization of the above metric.

(c) Redo Example 6.2 using the above metric.

6.2 (*AMI coding*) Alternate-mark-invert is a binary line coding scheme. The output signal is determined from the source's bit stream as follows:

- If the bit to be transmitted is a 0, then the signal is 0 volts over the bit period of T_b seconds.
- If the bit to be transmitted is a 1, then the signal is either $+V$ volts or $-V$ volts over the bit period of T_b seconds. It is $+V$ volts if previously a $-V$ volts was used to represent bit 1, $-V$ volts if previously a $+V$ volts was used to represent bit 1. Hence the name and mnemonic for the modulation.

(a) Draw the three waveforms and a signal space representation of the above modulation.

(b) Generally, the signal transmitted in any bit period depends on what happened previously. Thus there is memory and therefore a state diagram and a trellis. Draw a state diagram.

Label the transitions between the states with the input bit and the output signal.

(c) Now draw the trellis corresponding to the above state diagram. Start at $t = 0$ and assume that before $t = 0$ the voltage level corresponding to a 1 is $+V$ volts.

(d) Assume that the source bits are equally likely and that $V^2 T_b =$ 1 joule. Using the signal space diagram of (a) and trellis of (c) perform the Viterbi algorithm to demodulate the following set of outputs from the matched filter for the first 3 bit intervals:

$$r^{(1)} = 0.4;\ r^{(2)} = -0.8;\ r^{(3)} = 0.2 \quad \text{(volts)}. \qquad \text{(P6.2)}$$

6.3 (*NRZI coding*) Non-return-to-zero-inverse (NRZI) is a binary baseband signaling scheme used in magnetic recording. The NRZI signal waveform switches between the two amplitude levels of V volts and $-V$ volts as follows:

- The transitions from one amplitude level to another (V to $-V$ or $-V$ to V) occur only when the information bit is 1.
- No transition occurs when the information bit is a 0 (i.e., the amplitude level remains the same as the previous signal level).

(a) Draw the NRZI waveform for the information sequence {10110001}. Assume that the amplitude level before the transmission of the above sequence is $-V$ volts.

(b) Draw a state diagram. Label the transitions for the states with the input bit and the output signal.

(c) Now draw the trellis corresponding to the above state diagram. Start at $t = 0$ and assume that before $t = 0$ the voltage level is $-V$ volts.

(d) Assume that the source bits are equally likely and that $V^2 T_b =$ 1 joule. Draw the signal space diagram for the two signals used in NRZI. Then use this signal space diagram and the trellis of (c), sequence demodulate (with the Viterbi algorithm) the following set of outputs from the matched filter for the first 3 bit intervals:

$$r^{(1)} = 0.2;\ r^{(2)} = 0.6;\ r^{(3)} = -0.8 \quad \text{(volts)}.$$